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California Institute of Technology, Pasadena, California

DYNAMIC STABILITY OF BOMBS AND PROJECTILES

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6 September 1943

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Foreword

The four chapters comprising pages 1 to 123, inclusive, of this report were previously released in limited distribution with file designations and dates as follows:

Chapter I, CIT/JPC 4, January 2, 1943
Chapter II, CIT/JPC 5, January 2, 1943
Chapter III, CIT/JPC 6, January 2, 1943
Chapter IV, CIT/JPC 11, May 26, 1943

Chapters I and II were prepared as of July 1, 1942, Chapter III as of September 1, 1942, and Chapter IV as of the publication date. The text is herewith reproduced from the original vellum copy by the photo-offset process, with mathematical notation in the author's hand. The references listed at the end of Chapter IV were originally contained in CIT/JPC 11, and since this list includes all references cited in JPC 4, 5, and 6, the lists originally contained in these reports are here omitted.

The study as planned was intended to cover all aspects of the dynamic stability problems involved in solids moving through fluids. The chapters contained herein only partially cover the general subject. The results presented are immediately applicable to the ballistics of aerial bombs or the air trajectory of torpedoes. They also apply to the underwater ballistics of depth bombs and the underwater behavior of torpedoes in that phase of the trajectory where cavitation is absent.

Summary

Chapter I. Forces on a Solid Moving Through an Ideal Fluid.

The case of an ideal fluid is investigated as a first step in establishing theoretical expression for hydrodynamical forces. The inherent instability of elongated bodies moving end-on is discussed.

Chapter II. Stability Derivatives in a Real Fluid.

From equations for the forces in an ideal fluid, expressions are derived for the forces on streamlined bodies in a real fluid. Small yaw angles and absence of cavitation or compressibility effects are assumed. Wind- and water-tunnel measurements of the lift and drag factors affecting stability are discussed. The mechanism of the stabilizing action of fins is analyzed on the basis of the vortex theory.

Chapter III. Stability of the Rectilinear Trajectory in Air and Water Neglecting Gravity.

The stability of what might be designated as "free motion," where gravity is neglected, is investigated. The dependence of stability on the density of the projectile relative to the fluid is indicated. Application is made to the behavior of aerial bombs immediately after release from a plane. Certain equations are derived for perturbed motion, and the important influence of fluid density on stability is examined. The deviation of a trajectory caused by a perturbation of the bomb is analyzed; results for air and water are compared.

Chapter IV. Stability of the Vertical Fall.

The effect of gravity on stability is introduced and equations are derived for small perturbations of the trajectory about the vertical. This discussion differs from that in the previous chapter in the consideration of the possible existence of a critical velocity of fall. Results are applied to both aerial and underwater bombs, for cases limited to subsonic velocities in air and to bodies moving in water without cavitation.

General Conclusions

The density of a projectile relative to that of the fluid medium in which it is moving is an important factor in problems of dynamic stability. For bombs in air, stability in free motion is obtained provided the center of pressure is aft of the center of gravity; this condition is usually fulfilled by the use of large fins. The same projectile is more stable in water provided there is no cavitation.

Where the density of a projectile is very close to that of the fluid in which it is moving, little fin surface is required to maintain stability. This conclusion applies in the case of torpedoes in water, and explains why a torpedo launched from aircraft requires an additional stabilizer of considerable size to carry it through a regular air trajectory. The instability and scatter observed in practice as bombs enter the water is due largely to the existence of cavitation during the transition from air to water.

Instability of vertical fall in air may occur where bombs are released from rest or are released at very low velocities relative to the air. This instability appears in the form of oscillations of increasing amplitude with periods of approximately 2 to 3 sec; when the velocity has reached 100 to 300 ft/sec the instability vanishes and the oscillations become damped. This condition applies in the case of retro-bombing or the release of bombs from blimps and helicopters. In water such instability does not occur with bombs having elongated bodies with fins.

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DYNAMIC STABILITY OF BOMBS

AND PROJECTILES

by

M. A. BIOT

California Institute of Technology

Chapter I.

Forces on a Solid Moving Through an Ideal Fluid

CONTENTS

Introduction

In order to understand fully the nature of the fluid forces acting on a projectile it is important to analyse first the case of a solid moving through an ideal fluid. Although the results thus found do not yield the complete picture of the phenomena occurring in an actual fluid, it brings out in a simple form some of the essential features. The results in this chapter correspond to the case of pure irrotational potential flow about the solid. It is believed that the actual behavior of a viscous fluid can be described by superposing to this potential flow the so-called boundary layer effects and trailing vortices in pretty much the same way as in the wing theory. It must be kept in mind that this applies only to streamlined bodies and that cavitation or blunt nose effects must be excluded for the present. This matter will be treated in later chapters. Some important conclusions as regards the stability of streamlined bodies are already obtained from the restricted considerations of this chapter.

1.1 General Theory

Assume the fluid to be incompressible and frictionless. If the fluid surrounding the body is initially at rest the motion of the fluid is entirely due to that of the body and is therefore irrotational and acyclic. It is also assumed here that the fluid stays everywhere in contact with the body, i.e., there is no cavitation.

It is simple to conceive of the surrounding fluid as an addition to the inertia of the solid. However, these pure inertia resistances do not constitute the only forces acting on the solid. It was shown that even in an ideal fluid the moving solid experiences forces which depend also on its instantaneous velocity. Developments on this subject are due mainly to Sir W. Thompson, Tait and Kirchhoff during the middle of the last century *. The essential points of the procedure are summarized hereafter.

We choose a system of rectangular axis Ox, Oy, Oz fixed in the body and moving with it. These axes are sometimes referred to as "body axes". If the motion of the body at any instant be defined by the angular velocities p, q, r about these axes, and by the translational velocities u, v, w of the origin parallel to the instantaneous positions of these axes, the velocity at any point in the fluid is a linear function of u, v, w, p, q, r. The kinetic energy T of the fluid is therefore a homogeneous quadratic form which may be written,

$$\begin{aligned} 2T = {} & Au^2 + Bv^2 + Cw^2 + 2A'vw + 2B'wu + 2C'uv \\ & + Pp^2 + Qq^2 + Rr^2 + 2P'qr + 2Q'rp + 2R'pq \\ & + 2p(Fu + Gv + Hw) + 2q(F'u + G'v + H'w) + 2r(F''u + G''v + H''w) \end{aligned} \qquad (1.1)\ (1)$$

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* See H. Lamb - Hydrodynamics, Chap. VI. On the motion of solids through a liquid.

The 21 constants A, B, C, etc. are completely defined by the geometric configuration of the body, the position of the origin, and the orientation of the "body axes" with respect to the solid. It can be shown that the six components of the fluid impulse (momentum and moment of momentum) along the body axes are the six partial derivatives

$$\frac{\partial T}{\partial u}, \frac{\partial T}{\partial v}, \frac{\partial T}{\partial w}, \frac{\partial T}{\partial p}, \frac{\partial T}{\partial q}, \frac{\partial T}{\partial r}. \qquad (1.1)\ (2)$$

The fluid forces on the body are obtained by evaluating the rate of change of the fluid impulse with respect to fixed axes which coincide at the instant considered with the instantaneous position of the body axes. Denoting by X, Y, Z, and L, M, N, the components of these forces and of their moments along and about the body axes. It is found that

$$\begin{aligned}
-X &= \frac{d}{dt}\frac{\partial T}{\partial u} - r\frac{\partial T}{\partial v} + q\frac{\partial T}{\partial w} \\
-Y &= \frac{d}{dt}\frac{\partial T}{\partial v} - p\frac{\partial T}{\partial w} + r\frac{\partial T}{\partial u} \\
-Z &= \frac{d}{dt}\frac{\partial T}{\partial w} - q\frac{\partial T}{\partial u} + p\frac{\partial T}{\partial v} \\
-L &= \frac{d}{dt}\frac{\partial T}{\partial p} - w\frac{\partial T}{\partial v} + v\frac{\partial T}{\partial w} - r\frac{\partial T}{\partial q} + q\frac{\partial T}{\partial r} \\
-M &= \frac{d}{dt}\frac{\partial T}{\partial q} - u\frac{\partial T}{\partial w} + w\frac{\partial T}{\partial u} - p\frac{\partial T}{\partial r} + r\frac{\partial T}{\partial p} \\
-N &= \frac{d}{dt}\frac{\partial T}{\partial r} - v\frac{\partial T}{\partial u} + u\frac{\partial T}{\partial v} - q\frac{\partial T}{\partial p} + p\frac{\partial T}{\partial q}
\end{aligned} \qquad (1.1)\ (3)$$

The first term on the right side of these equations is the rate of change of the components of the fluid impulse. These are linear functions of the accelerations representing the apparent additional inertia of the body (see section 1.4) in the fluid. The remaining terms owe their existence to the fact that the coordinate axes move with the body. They correspond to forces which depend only on the instantaneous velocities. Also, the remarkable result is thereby established that all the forces depend only on the 21 coefficients in the expression of the kinetic energy.

The general formulae above will now be applied to the case of a body of revolution which leads to considerable simplification of the formulae and is of great practical importance for projectiles. Actually, it is not necessary that the body be of revolution. It can be shown that it will have the same dynamical properties as a body of revolution if there is an axis such that the body is superposed to itself when we rotate it through an angle $\frac{2\pi}{n}$ when n is an integer greater than 2. For instance, a body with fins at 120° or 90° will behave like a solid of revolution.

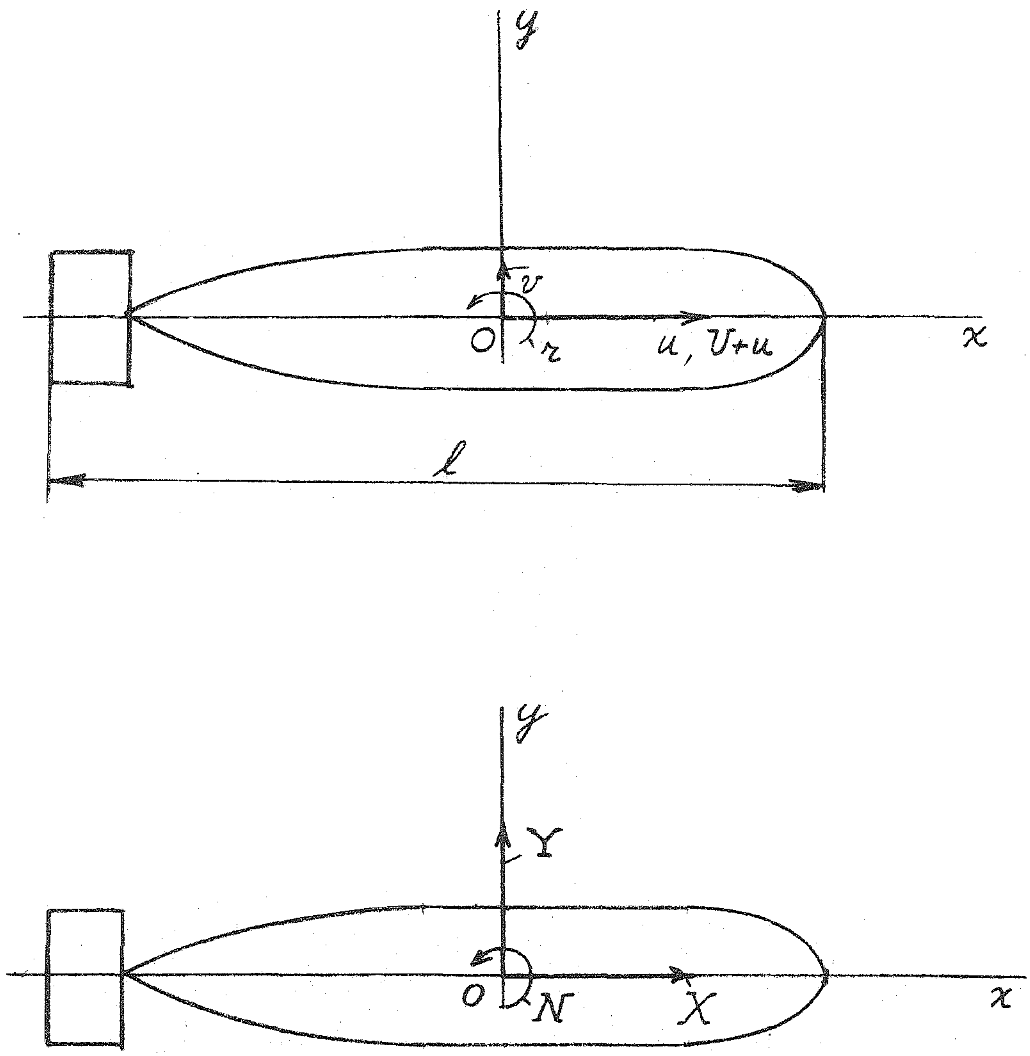

Figure. (1.2)(1)

1.2 The case of a solid of revolution

Choose the coordinate axis Ox to be along the axis of revolution of the body. The position of the origin O on the axis is left arbitrary for the present (figure (1.2) (1)). The solid is assumed to move in a fixed plane which coincides with the x, y plane attached to the body. The velocity of the origin O is projected on the body axes x, y with the components u, v and the angular velocity about Oz is denoted by r.

With $p = q = w = 0$, the expression (1.1)(1) for the kinetic energy of the fluid becomes

$$2T = Au^2 + Bv^2 + 2C'uv + Rr^2 + 2r(F''u + G''v) \qquad (1.2)(1)$$

A further simplification follows from the symmetry of the body since T must remain unchanged when v and r are replaced by −v and −r. This implies $C' = F'' = 0$.

Hence

$$2T = Au^2 + Bv^2 + Rr^2 + 2G''rv \qquad (1.2)(2)$$

Applying (1.1)(3) the forces exerted on the fluid by the body are

$$\begin{aligned} +X &= -\frac{d}{dt}\frac{\partial T}{\partial u} + r\frac{\partial T}{\partial v} \\ +Y &= -\frac{d}{dt}\frac{\partial T}{\partial v} - r\frac{\partial T}{\partial u} \\ +N &= -\frac{d}{dt}\frac{\partial T}{\partial r} + v\frac{\partial T}{\partial u} - u\frac{\partial T}{\partial v} \end{aligned} \qquad (1.2)(3)$$

or explicitly

$$\begin{aligned} X &= -A\dot{u} + r(Bv + G''r) \\ Y &= -B\dot{v} - G''\dot{r} - Aru \\ N &= -G''\dot{v} - R\dot{r} - (B-A)uv - G''ur \end{aligned} \qquad (1.2)(4)$$

1.3 Stability derivatives for small perturbations about rectilinear motion

We are particularly interested in the case where the body moves with a constant velocity U along its axis of symmetry. The problem is then to find the forces due to small perturbations of the velocity. To this purpose the velocity component u is replaced by $U + u$ and the quantities u, v, q are considered to be small of the first order compared to U. The products uv, uq, etc. are thus neglected as second order quantities. The expressions (1.2)(4) for the forces become

$$\begin{aligned} X &= -A\dot{u} \\ Y &= -B\dot{v} - G''\dot{r} - AUr \\ N &= -G''\dot{v} - R\dot{r} - (B-A)Uv - G''Ur \end{aligned} \qquad (1.3)(1)$$

The forces have now become linear functions of the velocity increments u, v, r, and their derivatives $\dot{u}$, $\dot{v}$, $\dot{r}$, with respect to time. The coefficients in these equations are called the stability derivatives. The terms involving $\dot{v}$, and $\dot{r}$ represent the inertia resistance of the fluid against acceleration. The coefficients in these terms are the so-called apparent fluid masses. Their significance will be discussed in section (1.4). Of the terms depending on the velocity components the most significant is $(B-A)Uv$ in the expression of the moment N. The physical meaning of this term is the object of section (1.6). The term AUr represents a transversal force due to angular velocity. The term $G''Ur$ does not correspond to an essential property of the forces since, as shown in the next section, it can be made to vanish by proper location of the origin O along the axis of the body.

Note that these forces are here shown to exist in an ideal fluid with pure potential flow. The velocity component v alone corresponds to what is known as a steady motion and produces a pure moment but no lift nor drag. This in accordance with general theorems and the so-called "d' Alembert's paradox" on the resistance of bodies in an ideal fluid. An angular velocity r on the other hand corresponds to an unsteady state of motion for which then general theorems do not apply. It is therefore no contradiction with these theorems that the angular velocity should produce a normal force component. For later use and also for the purpose of obtaining a clearer view of the physical significance of the above expressions it is worth while to investigate what happens to the forces when they are referred to <u>coordinate axes with fixed directions</u> instead of body axes. The origin O is again taken on the axis of symmetry and coincides with the origin of the body axes figure (1.3) (1) but the axes x' and y' have now fixed directions in space. The velocity components of O on x' and y' are denoted by $U+u'$, v' and the angle between x any x' by α. Note that $\dot{\alpha} = r$ is the angular velocity. The force components along the fixed directions are x' and y'. The transformation formulae are

$$\begin{aligned} X' &= X\cos\alpha - Y\sin\alpha \\ Y' &= X\sin\alpha + Y\cos\alpha \\ U+u &= (U+u')\cos\alpha + v'\sin\alpha \\ v &= -(U+u')\sin\alpha + v'\cos\alpha \end{aligned} \qquad (1.3)\ (2)$$

Assuming u u' v v' and α to be small these become

$$\begin{aligned} X' &= X - Y\alpha \\ Y' &= Y + X\alpha \\ u &= u' \\ v &= v' - U\alpha \end{aligned} \qquad (1.3)\ (3)$$

Substituting in (1.3) (1) and neglecting second order quantities

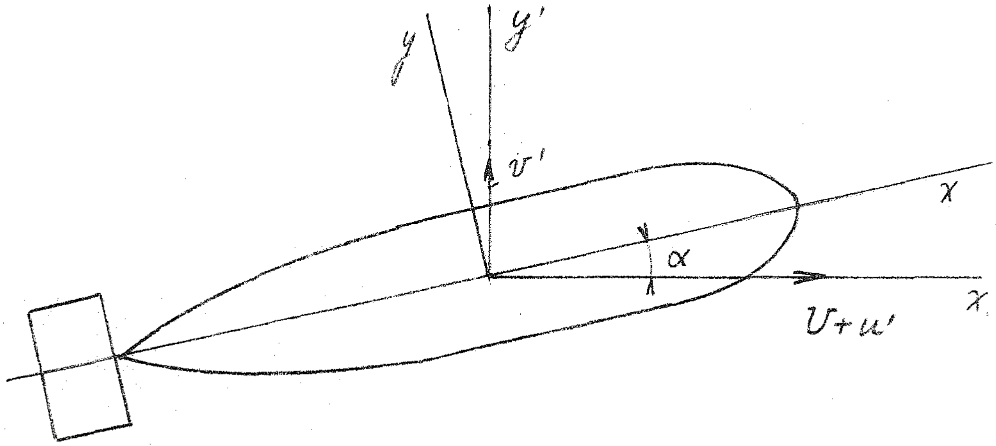

Figure (1.3)(1)

$$X' = -A\dot{u}'$$

$$Y' = -B\dot{v}' - G''\dot{r} + (B-A)Ur \qquad (1.3)\ (4)$$

$$N = -G''\dot{v}' - R\dot{r} - (B-A)Uv' + (B-A)U^2\alpha$$

In this form the components Y' N are identical with the lift and moment on a symmetric airfoil ~~in~~ performing small oscillations in a stream of velocity U provided the circulation is neglected.

Note that these expressions are quite different from the ones obtained previously (1.3) (1) using force and velocity components referred to body axes.

1.4 Apparent inertia and virtual center of mass of the fluid

It is seen that the coefficients of $\dot{v}$ and $\dot{r}$ represent the resistance of the fluid to acceleration. The resistance of the fluid to longitudinal acceleration (x direction) is equivalent to an increase of the mass of the body by an amount A. This mass A is called the longitudinal apparent mass of the fluid. Similarly, there is a transversal apparent mass B giving the resistance of the fluid to transversal acceleration ($\dot{v}$). For elongated bodies B is much larger than A. The coefficient R is equivalent to a moment of inertia and represents the inertia resistance of the fluid to an angular acceleration $\dot{r}$ about a transversal axis through the origin O. Important significance is attached to the coefficient G'' which represents an inertia coupling between $\dot{v}$ and the angular acceleration $\dot{r}$. It implies that a transversal acceleration $\dot{v}$ of the body produces an inertia moment $-G''\dot{v}$ about Oz and conversely an angular acceleration $\dot{r}$ about the same axis produces a transversal inertia resistance $-G''\dot{r}$. It is easy to verify that both G'' and R depend on the location of the origin O along the axis while A and B are independent of this location. Let us demonstrate this by using the expression (1.2) (2) for the kinetic energy of the fluid

$$2T = Au^2 + Bv^2 + Rr^2 + 2G''rv \qquad (1.4)\ (1)$$

Choosing a new origin O_1 at a distance x_1 from O (figure (1.4) (1)) the new expression for the transversal velocity v_1 is

$$v_1 = v + x_1 r \qquad (1.4)\ (2)$$

Substituting $v = v_1 - x_1 r$ in (1.4) (1)

$$2T = Au^2 + Bv_1^2 + (R - 2x_1 G'' + Bx_1^2)r^2 + 2(G'' - x_1 B) r v_1 \qquad (1.4)\ (3)$$

This may be written

$$2T = Au^2 + Bv_1^2 + R_1 r^2 + 2G_1'' r v_1 \qquad (1.4)\ (4)$$

with

$$R_1 = R - 2x_1 G'' + Bx_1^2$$
$$G_1'' = G'' - x_1 B \qquad (1.4)\ (5)$$

The new coefficient G_1'' may be put equal to zero by choosing O_1 such that

$$G'' - x_1 B = 0 \qquad (1.4)\ (6)$$

For such a point the fluid inertia produces no coupling between angular and transversal accelerations. This point will be referred to as the virtual center of mass of the fluid. The relation between R and R_1 becomes

$$R = R_1 + Bx_1^2 \qquad (1.4)\ (7)$$

i.e., the coefficient R follows the same law of transformation as the moment of inertia of a solid of mass B when the axis is moved to a distance x_1 from the center of gravity.

Note that if the origin of coordinate is chosen at the virtual center of mass, the forces on the body as expressed by (1.3) (1) become

$$X = -A\dot{u}$$
$$Y = -B\dot{v} - AUr \qquad (1.4)\ (8)$$
$$N = -R\dot{r} - (B-A)Uv$$

Aside from the inertia reactions $A\dot{u}$, $B\dot{v}$, and $R\dot{r}$ the forces on the body are reduced to a transverse force $-AUr$ acting at the critical center of mass and a couple expressed by $-(B-A)Uv$ when v is the transverse velocity of the critical center of mass.

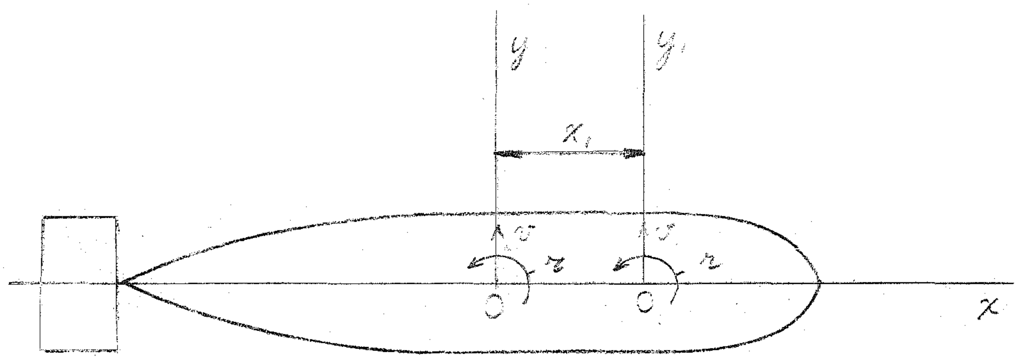

Fig (1.4) (-)

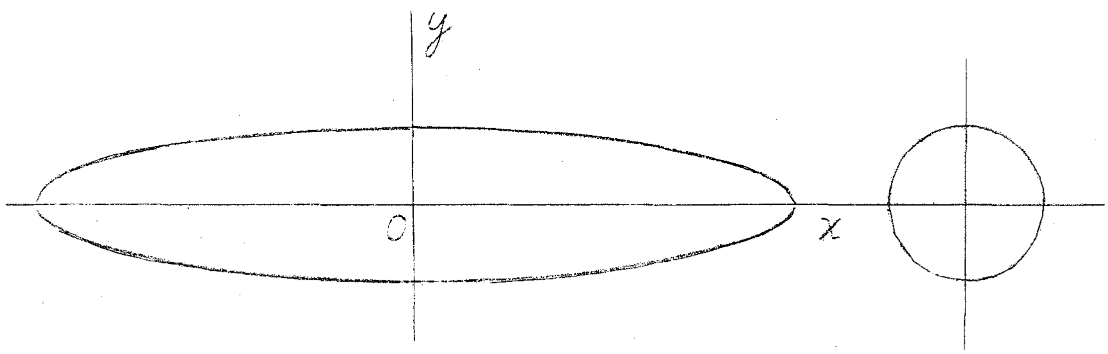

Figure (1.5)(1)

1.5 Inertia coefficients for a prolate ellipsoid

All the forces exerted on a solid by an ideal fluid are completely determined when the apparent masses are known. These may be evaluated from potential flow theory for bodies of simple geometrical shape.

Consider for instance the case of a prolate ellipsoid of revolution [figure (1.5) (1)]

The origin O is taken at the center which in this case is obviously the critical center of mass. The longitudinal and transversal apparent masses are written respectively in the form

$$A = \rho V k_1 \qquad (1.5)\ (1)$$
$$B = \rho V k_2$$

where ρ is the mass of the fluid per unit volume and V the volume of the ellipsoid. The so-called inertia coefficients k_1 and k_2 represent the additional mass as a fraction of the displaced mass of fluid. Similarly, the rotational apparent mass is written

$$R = I_f k' \qquad (1.5)\ (2)$$

when I_f is the moment of inertia of the displaced mass of fluid about a transversal axis through O.

The coefficients k_1, k , k^1 as computed by H. Lamb are given in table (1.5) (1) for various $\frac{\text{length}}{\text{diameter}}$ ratios of the ellipsoid.

Table (1.5) (1)

Inertia coefficients for a prolate ellipsoid

$\frac{\text{Length}}{\text{Diameter}}$	k_1	k_2	k^1	$k_2 - k_1$
Sphere 1	.5	.5	0	0
1.50	.305	.621	.094	.316
2.00	209	.702	.240	.493
2.51	156	.763	.367	.607
2.99	122	.803	.465	.681
3.99	082	.860	.608	.778
4.99	059	.895	.701	.836
6.01	045	.918	.764	.873
6.97	036	.933	.805	.897
8.01	029	.945	.840	.916
9.02	024	.954	.865	.930
9.97	021	.960	.883	.939
Cylinder	0	1.000	1.000	1.000

The case of a sphere corresponds to $\frac{\text{Length}}{\text{Diameter}} = 1$. In this case the transversal and longitudinal additional masses are the same and equal to one-half the mass of the displaced fluid. For an infinitely long circular cylinder the figure at the bottom of this table shows the transversal additional mass to be the same as that of the displaced fluid. The difference $k_2 - k_1$ is given in this table because of its importance in evaluating the couple on the body represented by the term $(B-A)Uv$ in equations (1.3) (1).

1.6 Inherent instability of a streamlined body

Assuming an elongated body moving end-on, the transversal mass B is larger than the longitudinal mass A. When this is the case, the term $(B - A)Uv$ represents a destablizing moment tending to turn the body to a positi n where it will move broadside-on. In other words the end-on motion is inherently unstable. This is easily shown by considering a fixed body immersed in a fluid stream of velocity components $-U$ and $-v$ along x and y [figure (1.6) (1)]. The axis of the body then lies at an angle $\alpha = \tan^{-1}\frac{v}{U}$ with the stream (angle of yaw). This case is identical with that of a body moving with the velocity components U, v in a fluid at rest. From formula (1.3) (1) there is a clockwise moment $(B - A)Uv$. For small values of α replacing v by $U\alpha$ this moment may be written $(B - A)U^2\alpha$. This is a pure couple proportional to the angle of yaw and the square of the velocity. It acts in a direction such that it tends to increase the angle of yaw. It is therefore a destablizing moment. The value of $(B - A)$ may be written

$$(B - A) = \rho V (k_2 - k_1) \qquad (1.6)\ (1)$$

when ρ is the fluid mass per unit volume, V the displaced volume of fluid. The coefficient $k_2 - k_1$ depends only on the shape of the body and is given for an elongated ellipsoid as a function of the $\frac{\text{length}}{\text{diameter}}$ ratio in table (1.5) (1). The table shows how fast the destablizing moment increases with increasing length when the volume is kept constant.

The existence of this destablizing moment is intrinsic property of elongated bodies moving end-on. In practice it has to be counteracted by the use of fins located at the tail. As pointed out previously by the writer (reference 21) a decrease in the destablizing moment can be expected

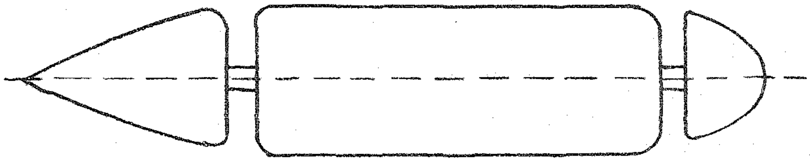

Figure (1.6)(2)

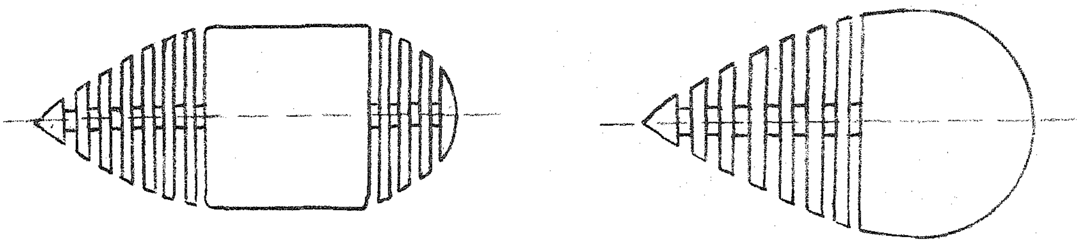

Figure (1.6)(3)

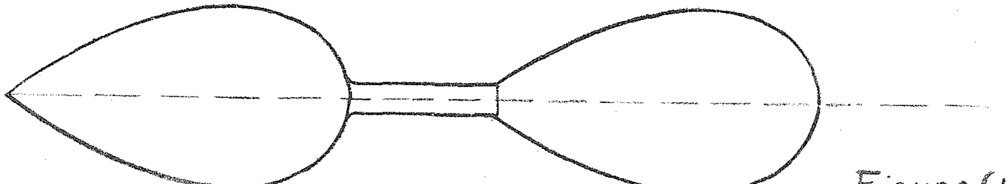

Figure (1.6)(4)

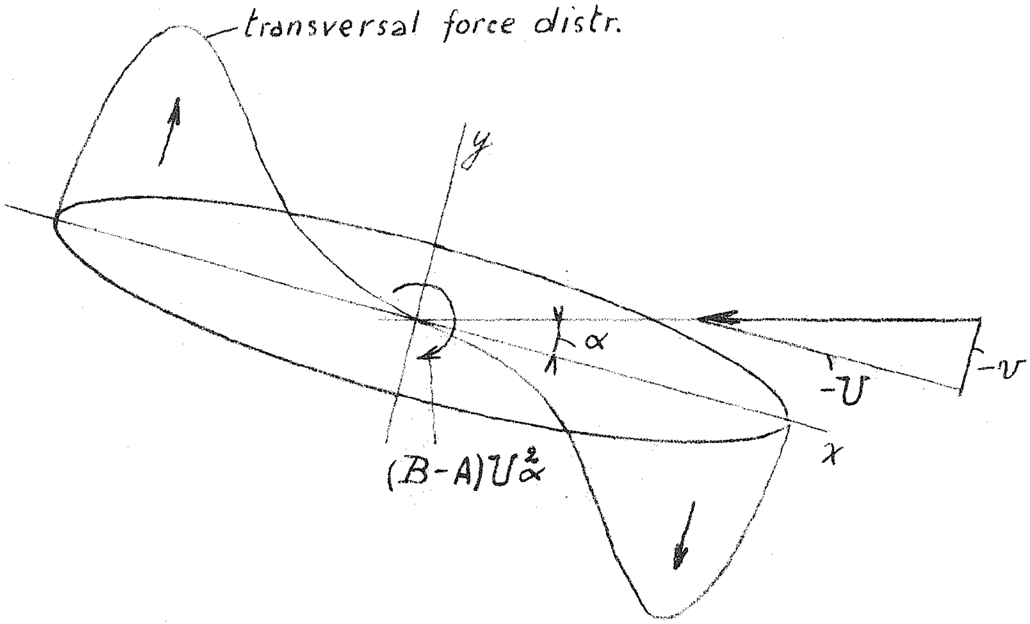

Figure (1.6) (1)

by the use of transversal slots in the body. This amounts to a decrease of the transversal apparent mass B. Figures (1.6) (2) and (1.6) (3) show several possible designs. The second figure shows cases of a body of length/diameter ratio approximately equal to unity which is streamlined by adding a pile of discs so as to produce a shape with transversal slots. Such a body should have its transversal and longitudinal mass pretty nearly equal ($B - A = 0$) and therefore should be nearly neutrally stable. Another method which applies the gap principle to the extreme is to separate the body into two or more tandem stubby streamlined shapes, as shown in figure (1.6) (4). This shape must have less instability than a single body of same volume and same total length.

Conclusions

The forces on a solid moving through an ideal fluid are determined as soon as we know the instantaneous kinetic energy of the fluid due to the velocity of the body. The case of a body of revolution is particularly simple. There exists a point on the axis referred to as the virtual center of mass (V.C) of the fluid. It can be considered as some kind of center of mass for the transversal apparent mass (B) of the fluid. There is also an apparent moment of inertia (R) of the fluid about a transversal axis through the V.C. An angular velocity r of the body produces a transversal force (AUr) applied at V.C. Finally a transversal velocity v of the V.C produces a pure destabilizing couple (B-A)Uv which is the origin of the inherent instability of streamlined bodies. This instability can be avoided or reduced by making B equal or nearly equal to A. Methods of obtaining this results by the use of transversal slots or separated bodies are indicated.

List of Symbols

x, y, z	coordinates along body axes (moving with the body)
u, v, w	velocity components of the origin parallel to the instantaneous position of x, y, z.
p, q, r	components of angular velocity parallel to the instantaneous position of x, y, z.
T	kinetic energy of the fluid due to the motion of the body
A	longitudinal apparent mass of the fluid (x direction)
B	transversal apparent mass of the fluid (y direction)
P	rotational apparent mass of the fluid about Oz
A' P, G'' etc.	coefficients in expression (1.1) (1)
X, Y, Z	components of the force excited on the body by the fluid parallel to x, y, z.
L, M, N.	moments of the forces exerted on the body by the fluid about x, y, z.
$\dot{u}$, $\dot{v}$, etc.	derivatives of u, v, etc. with respect to time.
U	large velocity component of the body in the x direction
α	angle of yaw (between x and x')
x' y'	axes with fixed directions.
X' Y'	force components of the fluid on the body parallel to the fixed directions x', y'.
u' v'	velocity components of the origin parallel to the fixed directions x', y'.
x_1	distance of virtual center of mass to the origin O.
v_1	velocity of the virtual center of mass parallel to y.
R_1	rotational apparent mass about the virtual center of mass.

List of Symbols (Cont)

k_1 k_2	inertia coefficient of the prolate ellipsoid for longitudinal and transversal apparent mass.
k'	inertia coefficient for the rotational apparent mass of the ellipsoid.
I_f	moment of inertia of the displaced fluid.
V	volume of displaced fluid.
ρ	mass of the fluid per unit volume.

DYNAMIC STABILITY OF BOMBS

AND PROJECTILES

By

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Chapter II.

Stability Derivatives in a Real Fluid

CONTENTS

Introduction

Having determined the forces on a body moving through an ideal fluid the next step is to establish expression for these forces in a viscous fluid. To this purpose the expression will be "linearized" by considering the case of a body of revolution moving in a direction near to its axis with small transversal and angular velocity perturbations. The problem then becomes one of determining the force increments due to this perturbation. The most general expression is obtained for these forces by taking them as linear functions of the perturbations. The characteristic coefficients thus introduced, called stability derivatives lead to the discussion of their mathematical nature, their dependence on the location of the origin and the orientation of the coordinate system. Methods of determining these coefficients experimentally are suggested. An attempt is also made to establish theoretical expressions for them by using the results obtained for an ideal fluid and adding the effect of fin lift and trailing vortices. This subject is only treated in embryonic form in this chapter. It is intended to develop this part of the theory more extensively later. Finally some numerical values of these coefficients from existing test data are given.

2.1 General expressions for the forces

Choose axes x, y attached to the projectile, x being directed along the axis of symmetry of the body. The origin O is left arbitrary for the present. As before, the projectile is assumed to move in a fixed plane which coincides with the x, y plane attached to the body (figure (2.1) (1)).

Denote by U+ u v the components of the velocity of point O on x and y, and by r the angular velocity about a transversal axis through O. Assuming the velocity increments u, v, r to be small with respect to U means that the projectile moves approximately on a straight line in the direction of its axis with a nearly constant velocity U and only slight perturbations from the uniform motion. Under these conditions, in addition to the drag D along x, the hydrodynamical forces on the body have a component Y along the y direction and a moment N about O (figure (2.1) (1)).

The forces D + X, Y, and moment N being due to the small perturbations u, v, and r may be expressed as linear functions of u, v, r and the accelerations $\dot{u}$, $\dot{v}$, $\dot{r}$. In the most general form this is written:

$$
\begin{aligned}
X &= -\rho V B_{uu}\dot{u} - \rho U \frac{V}{\ell} A_{uu} u - D \\
Y &= -\rho V(B_{vv}\dot{v} + B_{vr}\ell\dot{r}) + \rho U\frac{V}{\ell}(-A_{vv}v + A_{vr}\ell r) \qquad (2.1)(1) \\
N &= -\rho V\ell(B_{rv}\dot{v} + B_{rr}\ell\dot{r}) + \rho UV(A_{rv}v - A_{rr}\ell r)
\end{aligned}
$$

where ρ is the fluid mass per unit volume

V the volume of displaced fluid

ℓ the length of the projectile

The coefficients

$$
\begin{bmatrix} B_{uu} & 0 & 0 \\ 0 & B_{vv} & B_{vr} \\ 0 & B_{rv} & B_{rr} \end{bmatrix} \qquad \begin{bmatrix} -A_{uu} & 0 & 0 \\ 0 & -A_{vv} & A_{vr} \\ 0 & A_{rv} & -A_{rr} \end{bmatrix} \qquad (2.1)\ (2)
$$

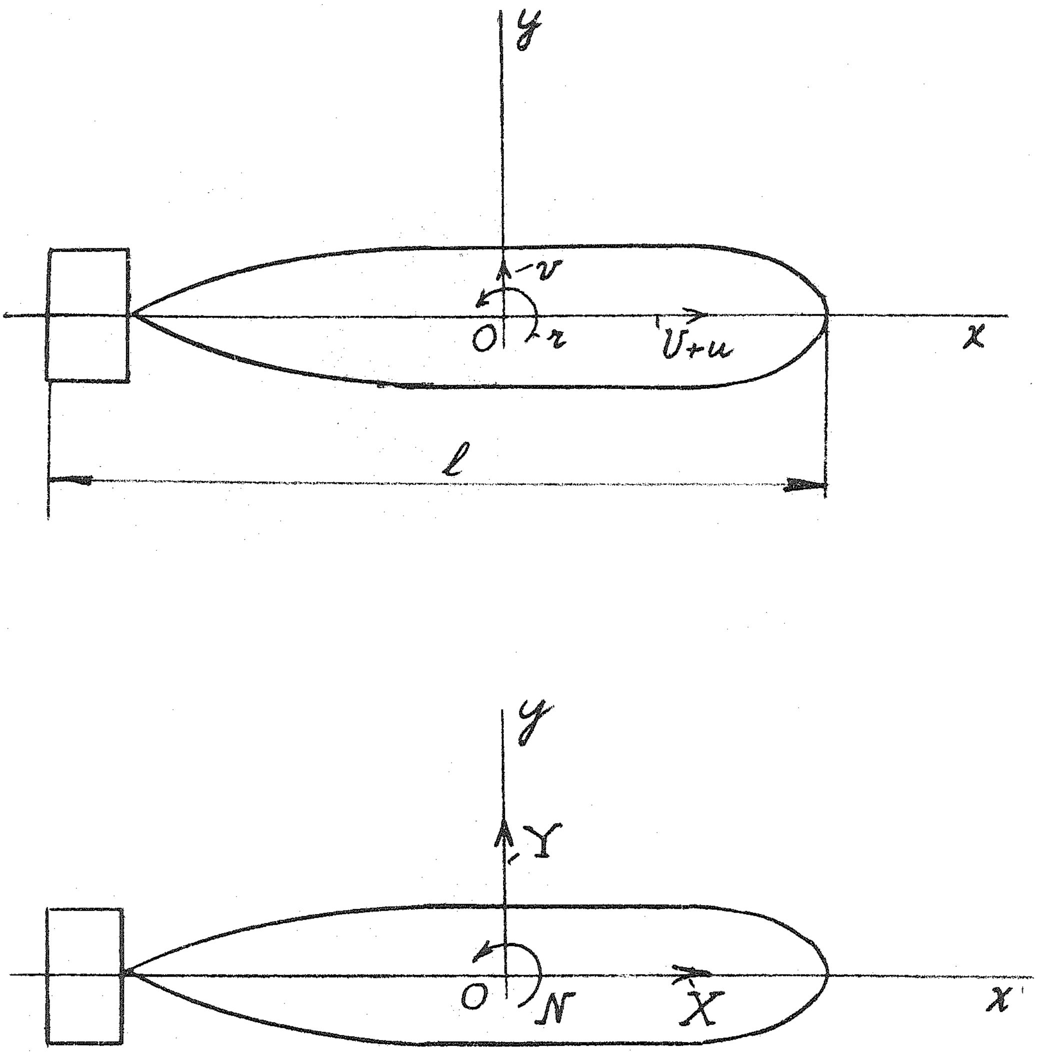


Figure (2.1)(1)

are non-dimensional. Noting that these coefficients represent essentially the increment of the forces on the body for an increment in the velocities and the accelerations they are referred to as stability derivatives.*

The above equations contain the implicit assumption that the force increments depend only on the velocities and accelerations of the projectile. Actually, of course, because of boundary layer and wake effects, the force increments depend on the complete history of the projectile and cannot be expressed rigorously in the form (2.1) (1). One might improve the accuracy by introducing derivatives of the velocities higher than the first. A better method seems to consider harmonic oscillations of the projectile, in which case the stability derivatives will turn out to be complex quantities depending on the frequency of oscillation. Actually, the coefficients will be found to depend on both the Reynolds number and the non-dimensional parameter $\frac{\omega \ell}{U}$ (ω circular frequency of oscillation). In the analogous case of the oscillating airfoil this parameter is called the reduced frequency. This procedure gives the complete picture of wake-hysteresis effect for any arbitrary perturbation of the projectile by the use of the Fourier analysis. It will be pointed out in section 2.5 how this hysteresis effect could be evaluated theoretically by the use of the vortex theory in a way entirely analogous to the treatment of an oscillating airfoil.

For the present we shall be contented to consider the stability derivatives to be dependent only on the Reynolds number. This must obviously be a good approximation for low values of the reduced frequency

The stability derivatives Buu, etc. will be referred to as the acceleration derivatives and Auu, etc. as the velocity derivatives.

---

* They might be referred to more exactly as the non-dimensional stability derivatives in order to distinguish them from the stability derivatives as defined in Chapter 1, section 1.3.

2.2 The acceleration derivatives as inertia coefficients

In expression (2.1) (1), consider first the terms depending on the accelerations. It will be assumed that these terms are identical with those computed for an ideal fluid - in Chapter 1. Comparing with the expression (1.3) (1), we find

$$\begin{aligned} \rho V B_{uu} &= A \\ \rho V B_{vv} &= B \\ \rho V l B_{vr} = \rho V l B_{rv} &= G'' \\ \rho V l^2 B_{rr} &= R \end{aligned} \qquad (2.2)\ (1)$$

The essential property of symmetry $B_{vr} = B_{rv}$ is thereby a direct consequence of the assumption just made.

When the body is a prolate ellipsoid of revolution with the origin at the center, these coefficients are readily found by using the table in Chapter I, section 1.5. We have

$$\begin{aligned} B_{uu} &= k_1 \qquad & B_{rv} = B_{vr} &= 0 \\ B_{vv} &= k_2 \qquad & B_{rr} &= \frac{I_f}{\rho V l^2} k' \end{aligned} \qquad (2.2)\ (2)$$

The coefficients B_{uu} and B_{vv} are respectively the inertia coefficients for longitudinal and transversal apparent mass. The coefficient B_{rr} represents the apparent mass for rotation, while $B_{vr} = B_{rv}$ is a measure of the distance to the origin of the "virtual center of mass" (see section 1.4) of the fluid.

In the case of a bomb, the following approximate procedure is suggested:

For the longitudinal mass, take the coefficient B_{uu} equal to k_1 for the ellipsoid of same length/diameter ratio. Consider the volume generated by the rotation of the body, including the fins, about the axis. We shall call this the virtual volume. Take the fluid displaced by this volume to be

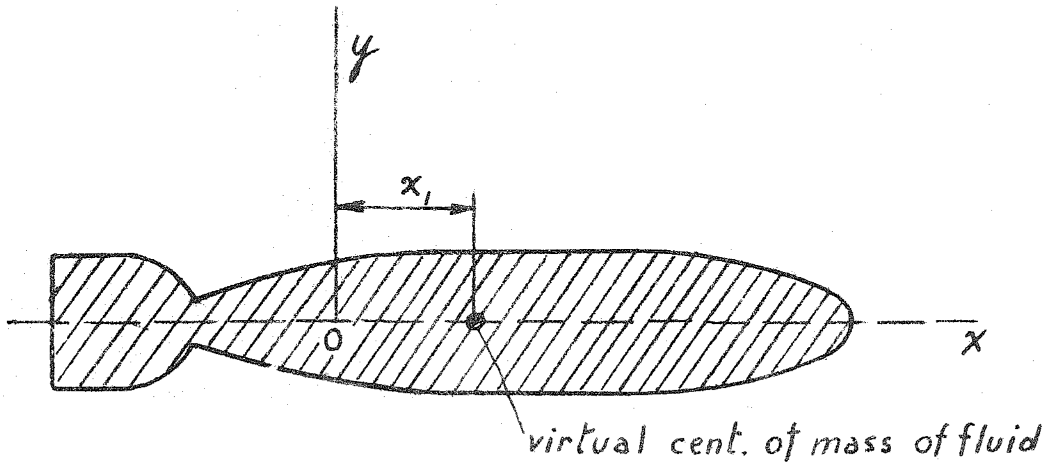


Figure (2.2)(1)

equal to the transversal mass B, and the virtual center of mass to be the center of mass of this virtual volume (figure (2.2) (1).

The coefficient $\mathcal{B}_{vr} = \mathcal{B}_{rv}$ is then given by (1.4) (6)

$$G'' = x_1 \mathcal{B} = \rho V \ell \mathcal{B}_{vr}$$

where x_1 is the distance of the virtual center of mass to the origin taken positive forward. Hence

$$\mathcal{B}_{vr} = \mathcal{B}_{rv} = \frac{x_1}{\ell} \frac{\mathcal{B}}{\rho V} \qquad (2.2)\ (3)$$

Finally, the coefficient $\mathcal{B}_{rr}$ is found from the relation

$$\rho V \ell^2 \mathcal{B}_{rr} = \mathcal{R} = I_f k' \qquad (2.2)\ (4)$$

$$\mathcal{B}_{rr} = \frac{I_f}{\rho V \ell^2} k'$$

The quantities in this expression are evaluated by putting I_f equal to the moment of inertia of the virtual volume of fluid about a transversal axis through the origin, and k' is a correction factor as given in Table (1.4) (1) for the ellipsoid of same $\frac{\text{length}}{\text{diameter}}$ ratio as the projectile.

It must be noted that the virtual volume is larger than the actual volume because in the rotation of the body about its axis, the fins generate a volume considerably larger than that of the fluid which they displace. Also, the virtual center of mass does not generally coincide with the center of buoyancy and will usually lie behind the latter; the distance increasing with increasing size of the fins.

From (2.2) (3) note also that the coefficient $\mathcal{B}_{vr}$ is approximately equal to $\frac{x_1}{\ell}$, being somewhat larger because $\mathcal{B} > \rho V$

It must be kept in mind that the above method of evaluating the acceleration derivatives holds only for streamlined bodies. Tests have shown that in this case the values found are correct at least in order of magnitude. (References 16, 17, 19). For blunt bodies or very stubby bodies, this obviously cannot be true in view of the thick boundary layer and large wake entrained by the body.

2.3 Dependence of the stability derivatives on the location of the origin of coordinates

The coefficients Buu, Auu etc. of equations (2.1) (1) are not invariant characteristics of the body but depend on the origin of coordinates. The question now is to find how these coefficients are transformed when the origin is moved to a point O" on the axis of symmetry at a distance εl of its previous position (figure (2.3) (1)).

The new force components X", Y", N" are:

$$\begin{aligned} X'' &= X \\ Y'' &= Y \\ N'' &= N - Y\varepsilon l \end{aligned} \qquad (2.3)\ (1)$$

The relation between velocity components are:

$$\begin{aligned} u &= u'' \\ v &= v'' - r''\varepsilon l \\ r &= r'' \end{aligned} \qquad (2.3)\ (2)$$

Substituting these values in equations (2.1) (1), it is found that they may be written:

$$\begin{aligned} X'' &= -\rho V B''_{uu}\dot{u}'' - \rho U\frac{V}{l}A''_{uu}u'' - D \\ Y'' &= -\rho V(B''_{vv}\dot{v}'' + B''_{vr}l\dot{r}'') + \rho U\frac{V}{l}(-A''_{vv}v'' + A''_{vr}lr'') \\ N'' &= -\rho Vl(B''_{rv}\dot{v}'' + B''_{rr}l\dot{r}'') + \rho UV(A''_{rv}v'' - A''_{rr}lr'') \end{aligned} \qquad (2.3)\ (3)$$

with the following values for the coefficients:

$$\begin{aligned} B''_{uu} &= B_{uu} \\ B''_{vv} &= B_{vv} \qquad & B''_{vr} &= B_{vr} - \varepsilon B_{vv} \\ B''_{rv} &= B_{rv} - \varepsilon B_{vv} \qquad & B''_{rr} &= B_{rr} - \varepsilon(B_{rv} + B_{vr}) + \varepsilon^2 B_{vv} \end{aligned} \qquad (2.3)\ (4)$$

and

$$\begin{aligned} A''_{uu} &= A_{uu} \\ A''_{vv} &= A_{vv} \qquad & A''_{vr} &= A_{vr} + \varepsilon A_{vv} \\ A''_{rv} &= A_{rv} + \varepsilon A_{vv} \qquad & A''_{rr} &= A_{rr} + \varepsilon(A_{rv} + A_{vr}) + \varepsilon^2 A_{vv} \end{aligned} \qquad (2.3)\ (5)$$

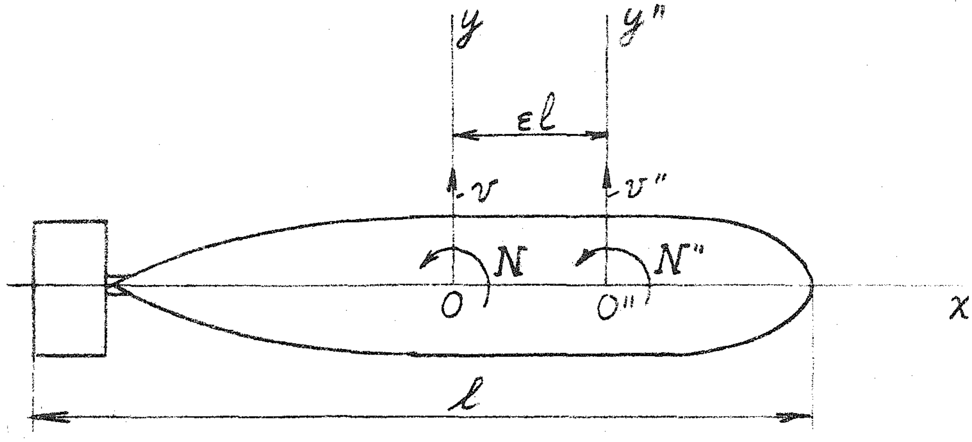


Figure (2.3)(1)

These are the transformation formulae for the stability derivatives. The B coefficients are seen to follow the transformation formulae already found for the inertia coefficients in section (1.4) when we assume $\mathcal{B}_{uv} = \mathcal{B}_{vu}$

The remarkable symmetry of these formulae is due to the fact that from the mathematical viewpoint, the stability coefficients are second rank tensors. Relations (2.3) (4) and (2.3) (5) are a particular case of the general transformation formulas for such quantities.

2.4 Method of measurement of the stability derivatives

Forces measured in a wind- or water tunnel are referred to axes with fixed directions, and the components are called lift and drag. For the purpose of measuring the stability derivatives, the first step is therefore to express the forces using axes with fixed directions (figure (2.4) (1)). Such axes were introduced already in section (1.3) and denoted by x' y'. Note that α is the angle between the fixed direction x' and the axis of the body and that $\dot{\alpha} = r$. The transformation formulae to be used are (1.3) (3). Using these formulae in expressions (2.1) (1) and neglecting second order quantities,

$$\begin{aligned} X' &= -\rho V B_{uu}\dot{u}' - \rho U \frac{V}{l} A_{uu} u' - D \\ Y' &= -\rho V (B_{vv}\dot{v}' + B_{vr} l\dot{r}) + \rho U \frac{V}{l}\left[-A_{vv}(v' - \alpha U) \right. \\ &\qquad \left. + (A_{vr} + B_{vv}) l r\right] - D\alpha \\ N &= -\rho V l (B_{rv}\dot{v}' + B_{rr} l\dot{r}) + \rho U V \left[A_{rv}(v' - \alpha U) \right. \\ &\qquad \left. - (A_{rr} - B_{rv}) l r\right] \end{aligned} \qquad (2.4)\ (1)$$

Consider now body to be immersed in a stream of velocity U and the origin O to be fixed. The body is assumed to rotate freely about the axis Oz (perpendicular to x' y'). In this case the velocity v' is zero, α denotes the angle of yaw of the body with the stream, and r the velocity of rotation about the axis Oz. The moment about this axis is then:

$$N = -\rho V l B_{rr} \dot{r} l - \rho U^2 V A_{rv}\alpha - \rho U V (A_{rr} - B_{rv}) l r \qquad (2.4)\ (2)$$

When the body has a fixed yaw angle α with the stream $r = \dot{r} = 0$, the moment reduces to: .

$$N = -\rho U^2 V A_{rv}\alpha \qquad (2.4)\ (3)$$

This yields the value of the coefficient A_{rv}.

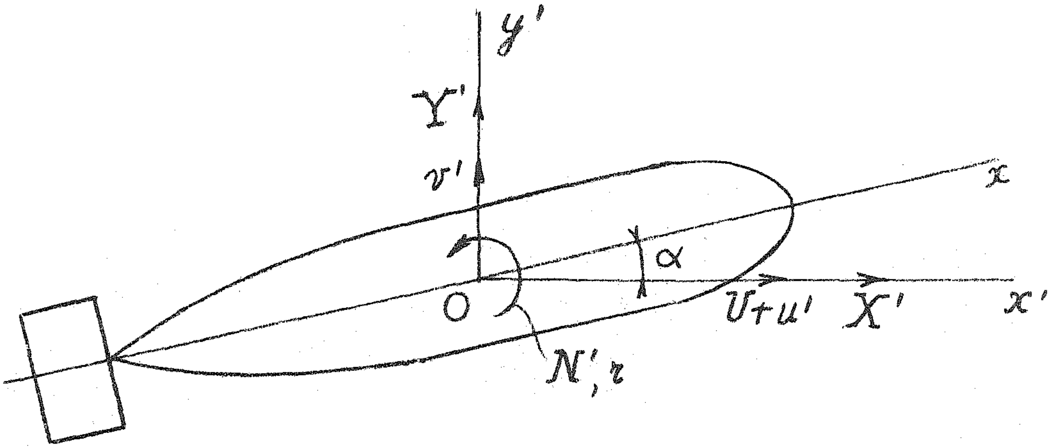


Figure (2.4)(1)

Assuming that the coefficients $\mathcal{B}_{rv}$ and $\mathcal{B}_{rr}$ have been determined theoretically, the coefficient A_{rr} may be evaluated by measuring the small oscillations of the body in the stream about the axis Oz. If A_{rv} is negative, it is necessary to add a restoring spring to the system in order to produce oscillations. Denoting by I_b the moment of inertia of the body about Oz and k the spring modulus, the equation for the oscillations of the body reads:

$$I_b \ddot{\alpha} + k\alpha = N \qquad (2.4)\ (4)$$

or

$$(I_b + \rho V \ell^2 \mathcal{B}_{rr})\ddot{\alpha} + \rho U V (A_{rr} - \mathcal{B}_{rv})\ell\dot{\alpha} + (k + \rho U^2 V A_{rv})\alpha = 0$$

The damping coefficient for the oscillation is:

$$\rho U V (A_{rr} - \mathcal{B}_{rv}) \qquad (2.4)\ (5)$$

Measurement of this motion yields the value of the coefficient A_{rr}

If this measurement is repeated with the oscillation axis going through a new origin O" at a distance $\varepsilon\ell$ in front of the previous one, it is possible to determine in the same way the coefficients A''_{rv} and A''_{rr}
Now using the transformation formulae (2.3) (5), we may write:

$$A''_{rv} = A_{rv} + \varepsilon A_{vv} \qquad (2.4)\ (5)$$

$$A''_{rr} = A_{rr} + \varepsilon(A_{rv} + A_{vr}) + \varepsilon^2 A_{vv}$$

Since A_{rv} and A_{rr} have been determined previously, these equations yield the values of A_{vv} and A_{vr}. The four velocity derivatives are thus completely determined.

A check can be made on the theoretical evaluations of $\mathcal{B}_{rv}$ by repeating this procedure with a third position for the axis of rotation. The values measured should be compatible with those already found.

Note that a measurement of the period of oscillation furnishes a method of evaluating $\mathcal{B}_{rr}$.

Also, if the tunnel balance provides a means of measuring not only moments but lift and drag, the second equation in (2.4) (1) may be used.

$$Y' = -\rho U^2 \frac{V}{\ell} A_{vv} \alpha - D\alpha \tag{2.4) (6}$$

This gives directly the coefficient A_{vv} by measuring Y' and D, and provides a further check on the correctness of the experimental procedure.

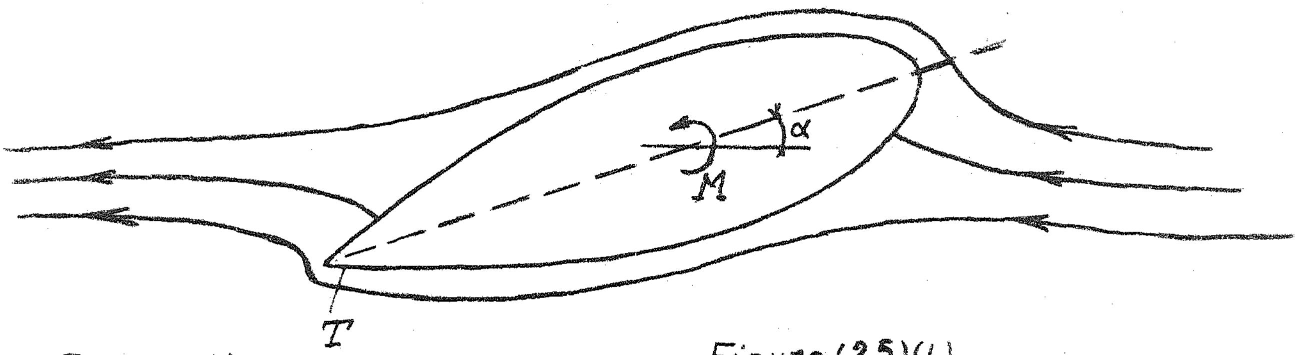
It may occur that the rotational motion of the body as represented by equation (2.4) (4) is highly damped or unstable so that the motion is not of the nature of a damped oscillation. This makes it difficult to evaluate the coefficients of the differential equation by the direct observation of frequency and decrement. When this difficulty cannot be avoided by a modification of either k or I_b or the introduction of some other known modification, it will be necessary to take a record of the time-angle relation when the body has been given an initial disturbance from equilibrium.

2.5 Theoretical Expressions for the Stability Derivatives

In order to estimate the velocity derivatives (A_{vv} A_{vr} etc) the assumption will be made that the only difference with the case of an ideal fluid lies in the action of the tail and fins. The stability derivatives will then be the same as those in equations (1.3)(1) except for a correction due to the action of the tail. This assumption is to some extent justified by the observations that potential flow usually exists around the nose of a streamlined body and that it breaks down only in regions adjacent to the after body.

The mechanism of such after body action is illustrated in figure (2.5)(1) and (2.5)(2). The first figure shows the potential flow lines about a bare streamlined hull at yaw angle α . The second figure shows how this flow is modified in an actual fluid. Due to the viscosity it is impossible for the flow to follow the potential pattern at the tail T. A pair of trailing vortices is generated at the after body in order to satisfy the condition of smooth flow around the tail. The existence of trailing vortices has been demonstrated by experiments on airship models reference 23

As shown in section (1.6) in the case of potential flow the forces consist of a pure destabilizing couple , M. Actually due to the creation of vorticity additional forces are originated mainly on the after body. These additional forces are represented by a resultant normal force Y_t [figure (2.5)(2)] This normal force is taken to be applied at a point T.C. on the axis somewhere in the after body and this point is referred to hereafter as the tail center. It must be kept in mind that Y_t does not represent the actual forces on the after body but only that part of the forces



POTENTIAL FLOW

Figure (2.5)(1)

ELEVATION

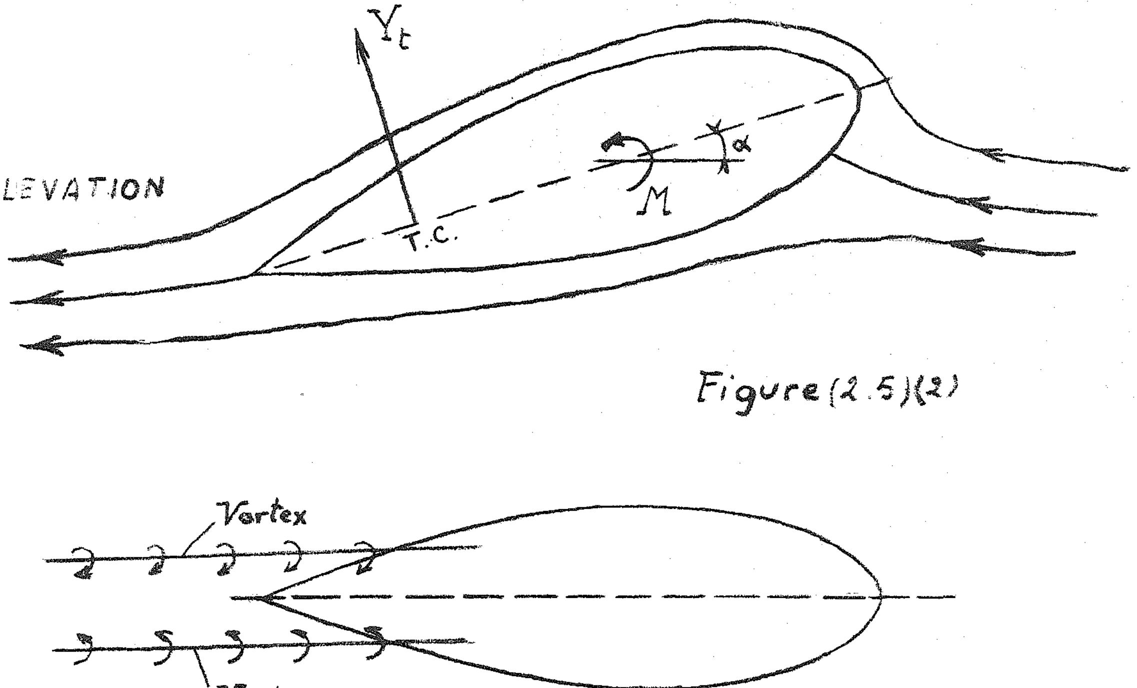


Figure (2.5)(2)

PLAN VIEW

due to the existence of vorticity in the wake. The existence of a lift is therefore associated essentially with the fact that vorticity is being shed from the after body.

The use of fins does not introduce any essentially new features in the mechanism just described but merely accentuates the effect by increasing the magnitude of the normal force Y_t and modifying the position of the tail center.

The question which now arises is how to express the value of Y_t when the body has both yaw and angular velocity. An assumption will be introduced here which finds some justification by its validity in the analogous case of an oscillating airfoil. It will be assumed that Y_t is proportion to the transversal velocity v_r of a certain point R.P on the axis of the body figure (2.5)(3). This point will be referred to as the rear point. The physical significance of this point is illustrated by the fact that if it is kept fixed while the body has an angular velocity about it no normal force Y_t is produced. It is a motion such that no vorticity is shed from the body and smooth potential flow still exists at the tail. The assumption just stated is expressed by the formula.

$$Y_t = \rho V \frac{U}{\ell} \tau v_r \qquad (2.5)(1)$$

The coefficient τ characterizes the intensity of the normal force or "lift". It is referred to as the tail lift factor.

The total forces on the body are obtained by adding those due to potential flow as expressed by (1.4)(8) and the normal force Y_t. Denoting by v_c the transversal velocity of the virtual center of mass, figure (2.5)(3), by $\ell\varepsilon_c$ (denoted previously by $-x_1$) the distance of the

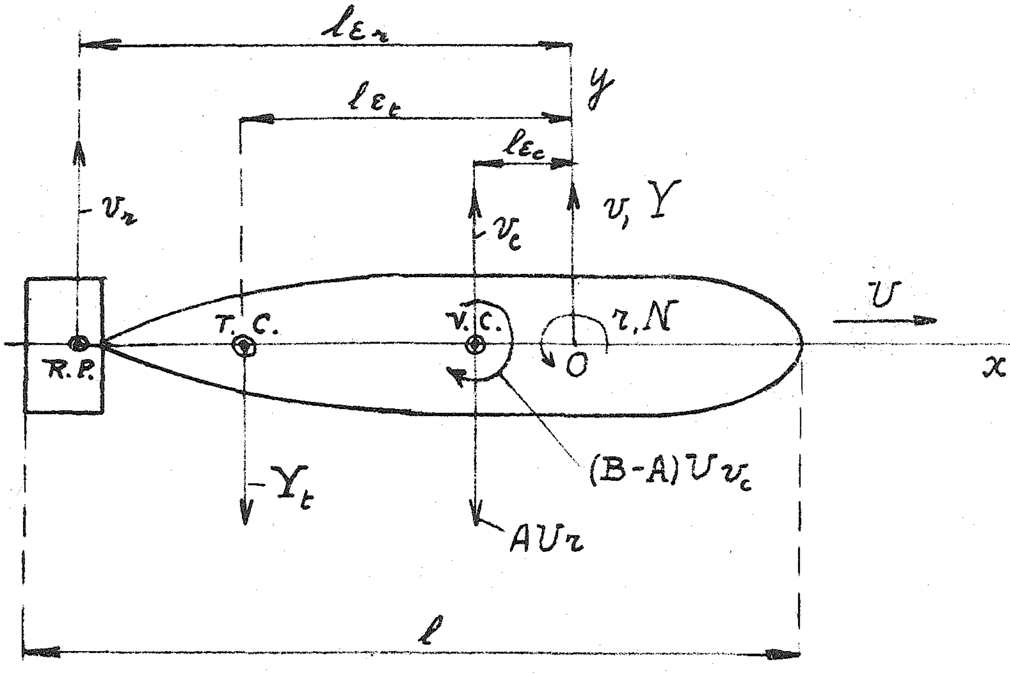

l_{ε_r}
l_{ε_t}
l_{ε_c}
y
v_r
v_c
v, Y
U
T. C.
V. C.
r, N
R. P.
O
x
Y_t
$(B-A)Uv_c$
AUr
l

Figure (2.5)(3)

origin O to the virtual center of mass and by $l\varepsilon_t$ the distance of the origin to the tail center we have

$$Y = -Y_t - AUr \tag{2.5)(2}$$

$$N = -(B-A)Uv_c + AU\varepsilon_c rl + Y_t l\varepsilon_t$$

N represents the moment about the origin. For simplicity the terms depending on the derivatives $\dot{v}$ and $\dot{r}$ have been omitted.

We substitute in these expressions the value (2.5)(1) and the kinematic relations

$$v_c = v - l\varepsilon_c r \tag{2.5)(3}$$

$$v_r = v - l\varepsilon_r r$$

where v denotes the transversal velocity of the origin and $l\varepsilon_r$ the distance of the rear point to the origin. We find

$$Y = \rho V \frac{U}{l}\left[-\tau v + (\tau\varepsilon_r - \frac{A}{\rho V}) rl\right] \tag{2.5)(4}$$

$$N = \rho V U\left[(\tau\varepsilon_t - \frac{B-A}{\rho V})v - (\tau\varepsilon_t\varepsilon_r - \frac{B}{\rho V}\varepsilon_c) rl\right]$$

Remembering that $B = \rho V B_{vv}$ and $A = \rho V B_{uu}$ the velocity derivatives are

$$A_{vv} = \tau \qquad\qquad A_{vr} = \tau\varepsilon_r - B_{uu} \tag{2.5)(6}$$

$$A_{rv} = \tau\varepsilon_t - (B_{vv} - B_{uu}) \qquad\qquad A_{rr} = \tau\varepsilon_t\varepsilon_r - B_{vv}\varepsilon_c$$

These give theoretical values for the velocity derivatives in terms of the location of the three characteristic points (R.P.) (T.C.) (V.C.) relative to the origin, the tail lift factor τ and the inertia coefficients B_{vv} and B_{uu}. It is apparent that the theory can be carried much further by introducing explicitly the concept of trailing vortices. When the body has a yaw angle with the stream a certain amount of lift is produced which is necessarily associated with a certain deviation of the flow in the wake known as a downwash. Due to the downwash, the effective angle of attack of the stream on the fins is smaller than the yaw angle of the body so that they will lose a certain amount of their effectiveness. This shadow effect will depend on the angle of yaw so that a non-linear relationship will be found between the forces and the yaw angle. The theory should follow very closely the vortex wing theory for the case of very low aspect ratios (reference 22). It should be possible to predict by this method to some extent the dependence of the stability derivatives on both the Reynolds number and the reduced frequency (sec. 2.1). Another important factor is the retarded flow in the wake due to boundary layer friction. If the fins lie in this retarded core their efficiency will be diminished. This seems to be the reason for using a circular fin lying outside the retarded wake.

Also from results obtained in the next section it seems that the modification induced by the fins on the flow around the body is in some cases far from negligible. This mechanism is well known in the airfoil theory where the action of ailerons produces most of its effect by modifying the flow about the wing.

The question of tail efficiency is therefore mainly a problem of mutual interference between the body and the fins. The above conclusions hold only for small angles of yaw. For values larger than the stalling angle it is clear that a quite different mechanism comes into play. These two phases of the flow pattern should be clearly separated in any attempt to predict the behavior of bombs.

2.6 Some Numerical Values of the Stability Derivatives

Some information can be obtained on the numerical values of the stability derivatives from existing static tests data of lift drag and moment of bodies. Much wind tunnel data is available on airship models and also some bomb data is available from tests in the C.I.T. high speed water tunnel.

In order to use this data the forces have to be expressed in the usual lift, drag, and moment, i.e., they must be referred to axes with fixed directions in space. The formulae for this transformation were already established in section (2.4). Limiting ourselves to the force Y' perpendicular to the stream velocity U and to the case of a fixed yaw angle α the formulae (2.4)(1) become

$$Y' = \left[\rho \frac{V}{l} U^2 A_{vv} - D\right]\alpha \qquad (2.6)(1)$$
$$N = -\rho V U^2 A_{xv} \alpha$$

In wind tunnel tests of airships the lift drag and moment are usually written in the form

$$Y' = c_L \frac{\rho U^2}{2} V^{2/3} \qquad (2.6)(2)$$
$$D = c_D \frac{\rho U^2}{2} V^{2/3}$$
$$N = c_m \frac{\rho U^2}{2} V^{2/3} l$$

where c_L = lift coefficient

c_D = drag coefficient

c_m = moment coefficient

Assuming a linear variation of c_L and c_m with small values of the yaw angle we may write

$$c_L = \frac{dc_L}{d\alpha}\alpha \qquad (2.6)(3)$$
$$c_m = \frac{dc_m}{d\alpha}\alpha$$

Comparing then (2.6)(1) and (2.6)(2) the following relations are found

$$\frac{1}{2}\left(\frac{dc_L}{d\alpha}+C_D\right)\frac{\ell}{V^{1/3}} = A_{vv}$$

$$\frac{1}{2}\,\frac{dc_m}{d\alpha}\,\frac{\ell}{V^{1/3}} = -A_{rv} \qquad (2.6)(4)$$

We take note that A_{vv} involves the measured of both the drag coefficients C_D and the so-called "slope of the lift curve" $\frac{dc_L}{d\alpha}$. The position of the tail center may be derived from expression (2.5)(6)

We have

$$\tau = A_{vv} \qquad \text{tail lift factor}$$

$$\tau\epsilon_t - B_{vv} - B_{uu} = A_{rv}$$

hence

$$\epsilon_t = \frac{1}{A_{vv}}\left[B_{vv} - B_{uu} + A_{rv}\right]$$

This gives the position of the tail center (T.C.). Values of these constants shall now be taken from wind tunnel data on airship models.

Four cases are considered.

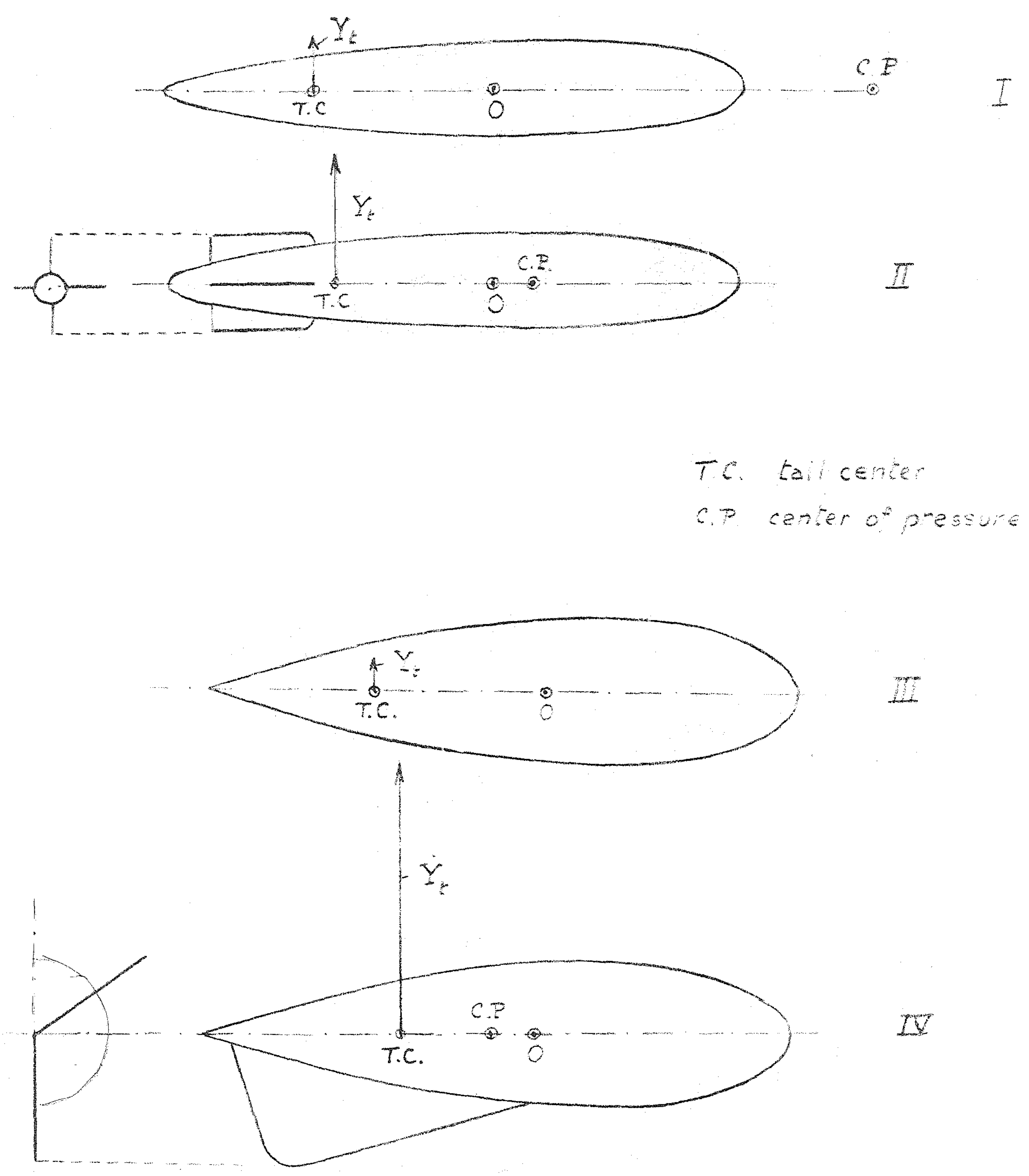
I. Bare hull of Goodyear Zeppelin as shown in figure (2.6)(1) I. The data for drag lift and moment is taken from references 11 and 22.

II. Same as case I with the addition of fins as shown in figure (2.6)(1) II.

III. Bare hull of kite balloon as shown in figure (2.6)(1) III. Data taken from references 20 and 22.

IV. Same as case III with the addition of plane fins as shown in figure (2.6)(1) IV.

Results are summarized in Table (2.6)(1).



Y_t
T.C
O
C.P
I
Y_t
T.C
O
C.P.
II
T.C. tail center
C.P center of pressure
Y_t
T.C.
O
III
Y_t
C.P
T.C.
O
IV

Figure (2.6)(1)

Table (2.6)(1)

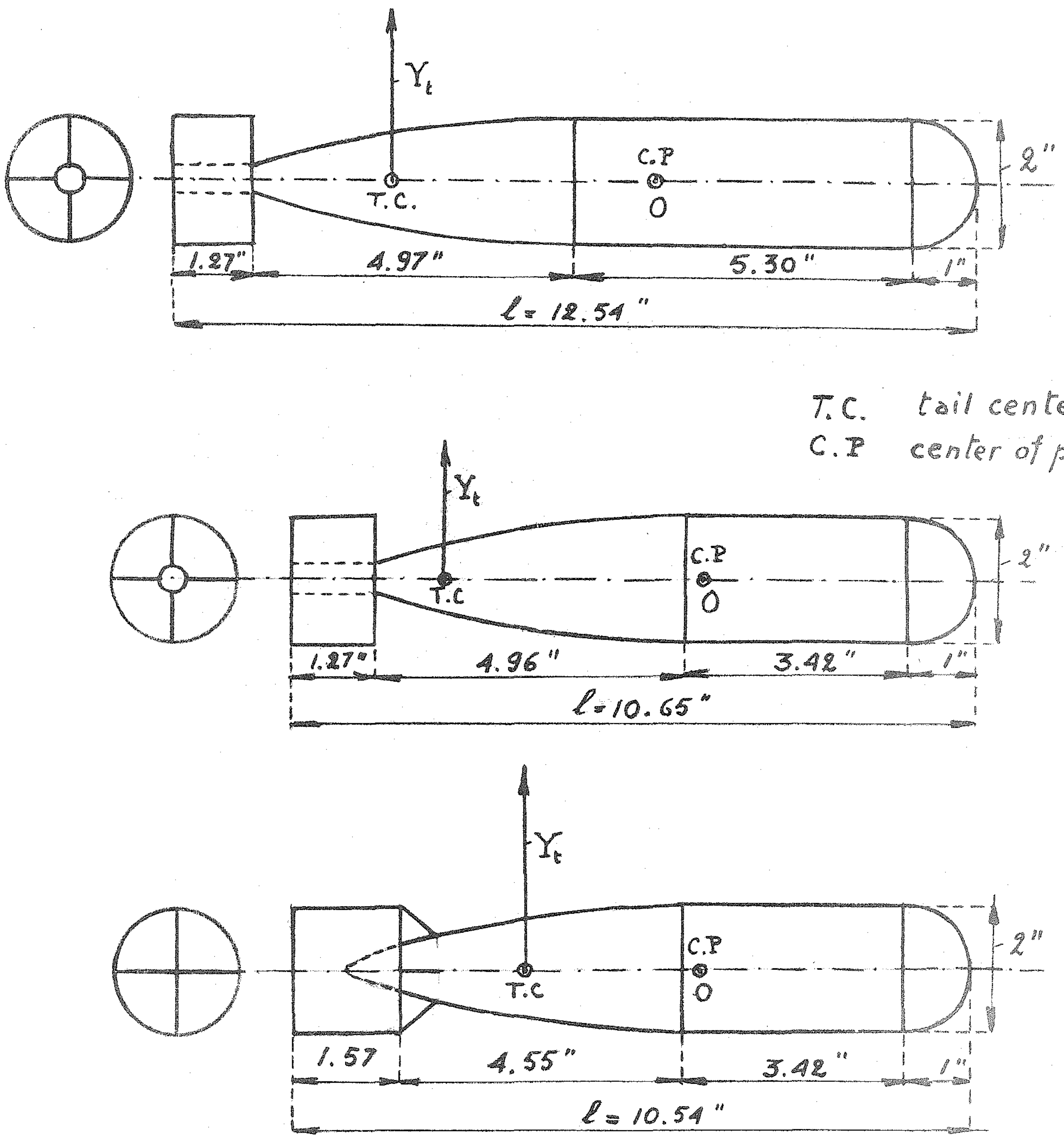
	$\frac{\ell}{V^{1/3}}$	$\frac{dc_L}{d\alpha}$	c_D	$\frac{dc_m}{d\alpha}$	$A_w = \tau$	A_{rv}	$B_w - B_{uu}$	ε_t	DISTANCE OF T.C. FROM NOSE
I	4	.445	.03	.29	.94	-.58	.87	.31	.74 ℓ
II	4	1.2	.04	.10	2.48	-.20	.87	.27	.69 ℓ
III	3.27	.237	.016	.397	.415	-.65	.77	.29	.73 ℓ
IV	3.27	2.88	.032	-.218	4.75	.35	.77	.23	.67 ℓ

In all these cases the origin of coordinates O is taken at the center of buoyancy so that ε_t denotes the distance of the center of buoyancy to the tail center. The position of the tail center is indicated as T.C. in the figures. The relative magnitudes of the normal force Y_t applied at T.C. is also indicated. Similar data was derived for bomb models from tests by R. T. Knapp in the high speed water tunnel at C.I.T. Dimensions of the bombs for the three cases considered are shown in figures (2.6)(2), I, II, III. For all these cases the moment is found to vanish about a point located at about 0.40 ℓ from the nose. This point is taken as origin O. The results are shown in table (2.6)(2).

Table (2.6)(2)

	A_w	$B_w - B_{uu}$	ε_t	Distance of T.C. from Nose
I	2.67	.873	.33	.73 ℓ
II	2.15	.830	.38	.78 ℓ
III	3.16	.830	.26	.66 ℓ

The origin O in all three cases is the point of application of the total force (resultant of the destabilizing moment and the normal force Y_t). This point is sometimes referred to as the center of pressure. (C.P.)



T.C. tail center
C.P center of pressure

Figure (2.6)(2)

The velocity of the water in these tests was around 20 to 30 ft./sec. and the Reynolds number based on the length of the order R.N. = 2.10^6. The contribution of the drag to the coefficient A_{vv} is small and amounts to 5%.

It will be noticed from the figures (2.6)(1) and (2.6)(2) that for both the wind-and water-tunnel tests the tail center lies quite a bit forward of the fins. This must be considered as an indication that in some cases the fins exert an appreciable action on the flow around the body. This point along with the shadow effect of the body on the fins should be submitted to further testing and correlated with theoretical consideration based on the vortex and boundary layer theory.

The position of the rear point cannot be determined from the above static tests. Test data on oscillating models, curved models or with the rotating arm should be used to obtain the derivatives A_{vr}, A_{rr} .
We will contend ourselves for the present to assume the location of the rear point in the applications to dynamic stability.

Conclusions

General expression has been established for the forces due to velocity perturbations of a projectile moving nearly on a rectilinear path along its axis. The stability derivatives are of two kind, the acceleration derivatives (coefficients of $\dot{v}$ and $\dot{r}$) and the velocity derivatives (coefficients of v and r). The acceleration derivatives are taken to be identical with the inertia coefficients for apparent mass introduced in Chapter I for potential flow. An approximate method of computing the position of the virtual center of mass is also given. The variation of the stability derivatives with the position of the origin of coordinates along the axis of the projectile is shown to follow simple transformation laws. The knowledge of these transformation laws is especially useful in deriving the value of the stability derivatives from wind-or water-tunnel measurements. It is shown that by using an oscillating model and making simple measurements with two different positions of the axis of oscillation the values of all four velocity derivatives may be derived. Theoretical expression for the stability derivatives are obtained by adding to the forces due to potential flow as obtained in Chapter I. those due to the generation of vorticity in the fluid at the afterbody.

These expressions depend essentially on the location of three characterictic points (R.P.) (T.C.) (V.C.) relative to the origin and a factor τ which is a measure of the amount of lift. It is apparent that the problem of tail efficiency lends itself to an extensive theoretical treatment by using the vortex theory in analogy with the case of airfoils. The downwash on the fins due to trailing vortices and also the fact that the boundary layer causes a wake of retarded flow both combine to decrease the fin efficiency. This explains the efficiency of ring fins. The position of the tail center derived from tests on airship and bomb models shows a remarkable tendency to lie appreciably ahead of the fins. This seems to indicate a considerable effect of the fins on the flow around the after body, thereby producing a lift on the body itself by interference.

M. A. Biot
July 1, 1942

List of Symbols

x, y, z	coordinates along body axes (moving with the body)
u, v, w	velocity components of the origin parallel to the instantaneous position of x, y, z.
p, q, r	components of angular velocity parallel to the instantaneous position of x, y, z.
T	kinetic energy of the fluid due to the motion of the body
A	longitudinal apparent mass of the fluid (x direction)
B	transversal apparent mass of the fluid (y direction)
R	rotational apparent mass of the fluid about OZ
A' F Q G'' etc.	coefficients in expression (1.1)(1).
X, Y, Z	components of the force excited on the body by the fluid parallel to x, y, z.
$L, M, N.$	moments of the forces exerted on the body by the fluid about x, y, z.
$\dot{u}, \dot{v}$ etc.	derivatives of u, v, etc. with respect to time.
U	large velocity component of the body in the x direction
α	angle of yaw (between x and x')
x' y'	axes with fixed directions.
X' Y'	force components of the fluid on the body parallel to the fixed directions x', y'.
u' v'	velocity components of the origin parallel to the fixed directions x', y'.
x_1	distance of virtual center of mass to the origin O.
v_1	velocity of the virtual center of mass parallel to y.
R_1	rotational apparent mass about the virtual center of mass.
k_1 k_2	inertia coefficient of the prolate ellipsoid for longitudinal and transversal apparent mass.
k'	inertia coefficient for the rotational apparent mass of the ellipsoid.
I_f	moment of inertia of virtual volume of fluid and moment of inertia of displaced fluid (in Chapter I)

V	volume of displaced fluid
ρ	mass of fluid per unit volume
ℓ	length of projectile
B_{vv} B_{uu} ... etc.	inertia coefficients (acceleration derivatives defined by (2.1)(1)
A_{vv} A_{uu} ... etc.	velocity derivatives defined by (2.1)(1)
D	drag
ω	circular frequency of oscillation
X" Y" N"	force components and moment of the fluid on the body when the origin is at O"
u" v" r"	velocity components when the origin is at O"
B''_{uu} B''_{vv} .... A''_{uu} A'_{vv}	stability derivatives when the origin is at O"
$\varepsilon \ell$	distance of O to O"
I_z	moment of inertia of the body about a transversal axis
k	spring constant
Y_t	transversal force due to tail action
$\ell\varepsilon_c$, $\ell\varepsilon_r$, $\ell\varepsilon_t$,	Distance of the origin to virtual center of mass (V.C.) rear point (R.P.) and tail center (T.C.) Figure (2.5)(3)
v_c	transversal velocity at V.C.
v_r	transversal velocity at R.P.
τ	tail lift factor
C_L C_D C_m	lift, drag, and moment coefficients for airship models.

DYNAMIC STABILITY OF BOMBS

AND PROJECTILES

by

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Chapter III.

Stability of the Rectilinear Trajectory in Air and Water Neglecting Gravity

CONTENTS

Introduction

The present chapter deals with the stability of a projectile moving in a fluid on a rectilinear trajectory at constant speed when the influence of gravity is neglected. The analysis is based on the expressions for the stability derivatives as derived in chapter II.

Spinning projectiles are not considered here and will be the object of a later treatment. It is assumed that if the projectile is moving in air the speed is below that at which shock phenomena appear and if moving in water there is no cavitation or air bubble around the body. Both these cases will be considered in later chapters.

The main purpose of these restrictions is to simplify the approach and bring out the dependence of the stability on the density of the projectile relative to the fluid; thereby investigating one of the main differences between aerial and underwater ballistics. The neglection of gravity is of course completely justified in the case of bodies where the gravity is cancelled by the buoyancy. The results are therefore of importance in predicting the behaviour of airships, submarines, or torpedoes. They should be particularly useful in the development of new underwater weapons belonging to the torpedo class.

After defining what is meant by stability the problem is to establish a simple stability criterion and to predict the amount of stability of various projectiles. It is also of importance to study deviations from the rectilinear trajectory due to initial perturbations in yaw and velocity of rotation. The general formulae derived hereafter for stability and perturbed trajectories are applied to examples of projectiles moving in air and water.

The effect of gravity on the stability is taken up in the next chapter.

3.1 Equations of motion and dynamic center

Consider a solid of revolution with body axes x, y, z. The axis of symmetry coincides with x and the motion is such that the x, y plane coincides with a fixed plane. This assumption is not restrictive when it is assumed that the motion is nearly rectilinear. The motion may then be considered as the superposition of two plane motions in perpendicular planes.

The mass of the body is denoted by m and the center of mass lies at a distance a from the origin o on the x axis. Figure (3.1)(1) We shall assume that the total force along the x axis is zero so that the velocity U along the axis is constant. The variable velocity components are then the component v along y and the angular velocity r about oz.

To write the equations of motion consider first the linear acceleration of the C.G. of the body and the angular acceleration about this point. Because of the angular velocity r of the coordinate system the y component of the C.G. acceleration is expressed by

$$m(\dot{v}_g + Ur) \qquad (3.1)(1)$$

where v_g is the y component of the C.G. velocity. Also the moment of the inertia forces about the C.G. is

$$\bar{I}_b \dot{r} \qquad (3.1)(2)$$

where $\bar{I}_b$ is the moment of inertia of the solid about a transversal axis through the C.G.

If N denotes the moment of the fluid forces about the origin and Y the y component of these forces then $N - aY$ is the moment of these forces about the C.G. Hence the two equations of motion

$$\begin{aligned} m(\dot{v}_g + Ur) &= Y \\ \bar{I}_b \dot{r} &= N - aY \end{aligned} \qquad (3.1)(3)$$

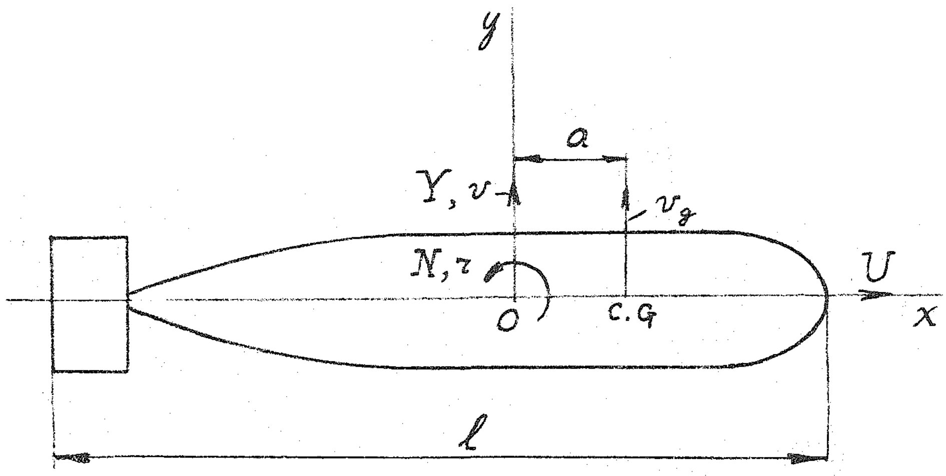


Figure (3.1)(1)

Substituting the value of Y in the second equation

$$m(\dot{v}_g + Ur) = Y$$
$$\bar{I}_b \dot{r} + am(\dot{v}_g + Ur) = N \qquad (3.1)\ (4)$$

These equations may still be transformed by introducing the moment of inertia I_b of the body about a transversal axis through the origin and the component v of the velocity of the origin. We have,

$$I_b = \bar{I}_b + ma^2$$
$$v_g = v + ar \qquad (3.1)\ (5)$$

Substituting in (3.1)(4) we finally get

$$m\dot{v} + ma\dot{r} + mUr = Y$$
$$ma\dot{v} + I_b\dot{r} + maUr = N \qquad (3.1)\ (6)$$

Expressions for the fluid forces Y and N are now introduced according to (2.1)(1) of chapter II

$$Y = -\rho V(B_{vv}\dot{v} + B_{vr}\dot{r}\ell) + \rho U \frac{V}{\ell}(-A_{vv}v + A_{vr}r\ell)$$
$$N = -\rho V\ell(B_{rv}\dot{v} + B_{rr}\dot{r}\ell) + \rho UV(A_{rv}v - A_{rr}r\ell) \qquad (3.1)\ (7)$$

Substituting in (3.1)(6) these equations become

$$\left(\frac{m}{\rho V} + B_{vv}\right)\dot{v} + \left(\frac{m}{\rho V}\frac{a}{\ell} + B_{vr}\right)\dot{r}\ell = \frac{U}{\ell}\left[-A_{vv}v + \left(A_{vr} - \frac{m}{\rho V}\right)r\ell\right]$$
$$\left(\frac{m}{\rho V}\frac{a}{\ell} + B_{rv}\right)\dot{v} + \left(\frac{I_b}{\rho V\ell^2} + B_{rr}\right)\dot{r}\ell = \frac{U}{\ell}\left[A_{rv}v - \left(A_{rr} + \frac{m}{\rho V}\frac{a}{\ell}\right)r\ell\right] \qquad (3.1)(8)$$

These are the general equations of motion with an arbitrary point along the axis chosen as origin. Since $B_{rv} = B_{vr}$ it is seen that it is possible to simplify the equations by choosing the location of the origin such that

$$\frac{ma}{\rho V\ell} + B_{vr} = \frac{ma}{\rho V\ell} + B_{rv} = 0 \qquad (3.1)\ (9)$$

This determines the distance a of the origin to the C.G. of the body. The equations of motion then become

$$\left(\frac{m}{\rho V} + B_{vv}\right)\frac{\dot{v}\ell}{U} = -A_{vv}v + \left(A_{vr} - \frac{m}{\rho V}\right)r\ell$$
$$\left(\frac{I_b}{\rho V\ell^2} + B_{rr}\right)\frac{\dot{r}\ell^2}{U} = A_{rv}v - (A_{rr} - B_{vr})r\ell \qquad (3.1)\ (10)$$

The point at which the origin must be located in order that equations (3.1)(9) be satisfied is referred to herein as the dynamic center. The physical significance of this point is readily grasped if we introduce the virtual center of mass (at a distance x_1 from o) and the transversal apparent mass $\mathcal{B}$. From (2.2)(3) of chapter II,

$$\mathcal{B}_{vr} = \mathcal{B}_{rv} - \frac{x_1}{l} \frac{\mathcal{B}}{\rho V} \qquad (3.1)(11)$$

Substituting in (3.1)(9)

$$ma + x_1 \mathcal{B} = 0 \qquad (3.1)(12)$$

This equation expresses the property that the dynamic center is the C.G. of the combination of the body mass m and the transversal apparent mass B the latter being concentrated at the virtual center of mass. The dynamic center is also defined physically by the property that a transversal force applied to the immersed body at that point does not produce any angular acceleration of the body.

Putting $-\frac{x_1}{l} = \varepsilon_c$ in accordance with the notation of chapter II, section (2.5), and remembering that $\frac{\mathcal{B}}{\rho V} = \mathcal{B}_{vv}$

$$\mathcal{B}_{vr} = \mathcal{B}_{rv} = -\varepsilon_c \mathcal{B}_{vv} \qquad (3.1)(13)$$

Further simplification is obtained by introducing a non dimensional variable

$$s = \frac{Ut}{l} \qquad (3.1)(14)$$

instead of the time t. The derivative with respect to s is related to the time derivative by the relation

$$\frac{d}{ds} = \frac{l}{U}\frac{d}{dt} \qquad (3.1)(15)$$

Using (3.1)(13) and the variable s equations (3.1)(10) are further simplified to

$$\left(\frac{m}{\rho V} + B_{vv}\right)\frac{dv}{ds} = -A_{vv}\,v + \left(A_{vr} - \frac{m}{\rho V}\right) r\ell$$

$$\left(\frac{I_b}{\rho V \ell^2} + B_{rr}\right)\frac{d(r\ell)}{ds} = A_{rv}\,v - \left(A_{rr} + \varepsilon_c B_{vv}\right) r\ell \qquad (3.1)(16)$$

Note that the new variable s is proportional to the time t and represents the distance through which the body has travelled after a time t if the length of the body is taken as unity.

Finally the following notation is introduced

$\frac{m}{\rho V} = \mu$ density of body relative to the fluid

$I_b = mk^2$

k radius of gyration of body relative to the dynamic center.

$\mu + B_{vv} = \mu_y$ transversal density

$\mu \frac{k^2}{\ell^2} + B_{rr} = \mu_y i^2$

i a radius of gyration expressed as a fraction of the length ℓ for the total transversal mass (body and fluid) about the dynamic center

$A_{vv} = C_{vv}$

$A_{vr} - \mu = C_{vr}$

$A_{rv} = C_{rv}$

$A_{rr} + \varepsilon_c B_{vv} = C_{rr}$

With these notations the equations of motion are written

$$\mu_y \frac{dv}{ds} = -C_{vv}\,v + C_{vr}\,r\ell$$

$$\mu_y i^2 \frac{d(r\ell)}{ds} = C_{rv}\,v - C_{rr}\,r\ell \qquad (3.1)(17)$$

Another simplification, which is perhaps not quite essential but reduces the writing in further applications, results from using the non dimensional unknowns

$$v^* = \frac{v}{U} \qquad r^* = \frac{rl}{U}$$

$$\mu_y \frac{dv^*}{ds} = -C_{vv} v^* + C_{vr} r^*$$

$$\mu_y i^2 \frac{dr^*}{ds} = C_{rv} v^* - C_{rr} r^* \qquad (3.1)(18)$$

It is worth noting that the simplified equations contain non dimensional parameters and therefore express explicitly the laws of dynamic similarity for the motion. The velocity U does not appear explicitly in the equations. It is however contained in the coefficients C insofar as the velocity derivatives are taken to vary with the Reynolds number.

It was shown in Chapter II (2.5)(6) that the velocity derivatives A_{vv} A_{rv} etc. can be expressed as

$$A_{vv} = \tau \qquad A_{vr} = \tau\varepsilon_r - B_{uu}$$

$$A_{rv} = \tau\varepsilon_t - (B_{vv} - B_{uu}) \qquad A_{rr} = \tau\varepsilon_t\varepsilon_r - B_{vv}\varepsilon_c \qquad (3.1)(19)$$

Since the origin is now the dynamic center the quantities $\varepsilon_r, \varepsilon_t, \varepsilon_c$, express the distance respectively of the rear point, the tail center, and the virtual center of mass (as fraction of l) to the dynamic center.

Also the coefficients in the equations (3.1)(18) are

$$C_{vv} = \tau \qquad C_{vr} = \tau\varepsilon_r - B_{uu} - \mu$$

$$C_{rv} = \tau\varepsilon_t - (B_{vv} - B_{uu}) \qquad C_{rr} = \tau\varepsilon_t\varepsilon_r \qquad (3.1)(20)$$

3.2 Definition and Criterion of stability

The meaning of dynamic stability is formulated by the question: if a projectile in rectilinear motion is given a perturbation will it go back to a rectilinear motion?

The stability of rectilinear motion is entirely determined by the equations of motion (3.1)(18)

$$\mu_y \frac{dv^*}{ds} = -C_{vv} v^* + C_{vr} r^*$$
$$\mu_y i^2 \frac{dr^*}{ds} = C_{rv} v^* - C_{rr} r^* \qquad (3.2)(1)$$

To answer this question, eliminate v and $\frac{dv}{ds}$ between the two equations (3.2)(1); this yields an equation for r^*,

$$\mu_y^2 i^2 \frac{d^2 r^*}{ds^2} + \mu_y E_1 \frac{dr^*}{ds} + E_2 r^* = 0 \qquad (3.2)(2)$$

where

$$E_1 = C_{vv} i^2 + C_{rr} = \tau(i^2 + \varepsilon_t \varepsilon_r)$$
$$E_2 = C_{vv} C_{rr} - C_{rv} C_{vr} = (B_{vv} - B_{uu})\tau\varepsilon_r + (\tau\varepsilon_t - B_{vv} + B_{uu})(B_{uu} + \mu) \qquad (3.2)(3)$$

The general solution of equation (3.2)(2) is

$$r^* = C_1 e^{\lambda_1 s} + C_2 e^{\lambda_2 s} \qquad (3.2)(4)$$

with

$$\lambda_1 = \frac{1}{2\mu_y i^2}\left[-E_1 + \sqrt{E_1^2 - 4i^2 E_2}\right]$$
$$\lambda_2 = \frac{1}{2\mu_y i^2}\left[-E_1 - \sqrt{E_1^2 - 4i^2 E_2}\right] \qquad (3.2)(5)$$

The condition for the motion to become uniform and rectilinear after a perturbation is that the angular velocity r returns to zero after a sufficient length of time. Hence the exponentials in (3.2)(4) must tend to zero for $s \to \infty$ Therefore λ_1 and λ_2 must be both negative or if imaginary their real part must be negative. This is expressed by the conditions

$$E_1 > 0 \qquad E_2 > 0$$

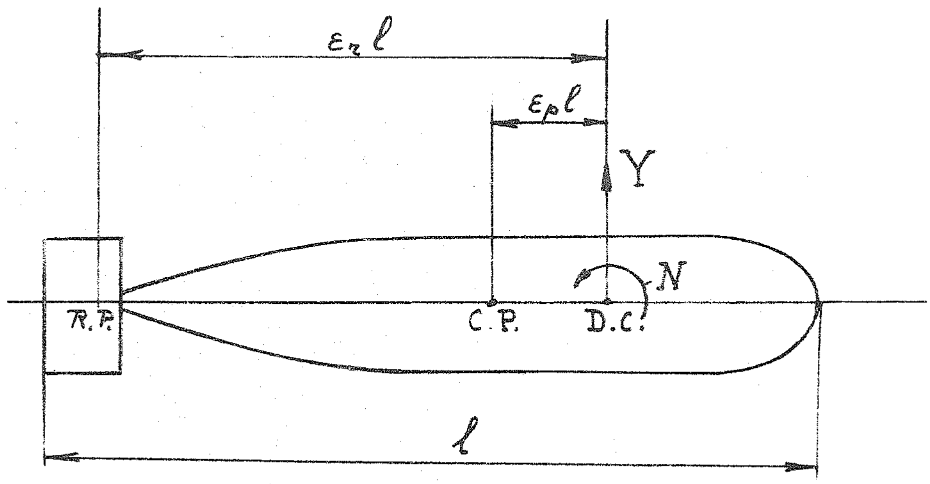


Figure (3.2)(1)

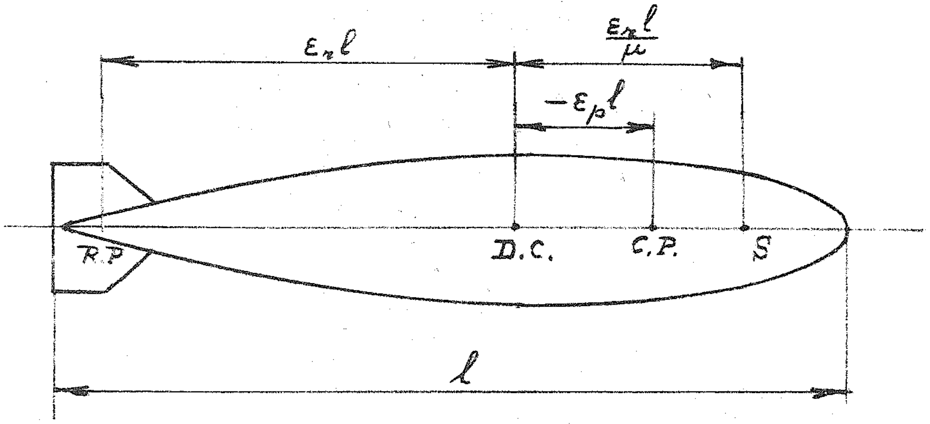


Figure (3.2)(2)

These conditions also follow from the physical interpretation of equation (3.2)(2) which may be taken to represent the oscillation of a damped spring mass system with a spring constant E_2 a damping factor $\mu_y E_1$ and a mass $\mu_y^2 i^2$. In order for the motion to be stable both the spring constant and the damping factor must be positive.

Introducing now the theoretical values (3.2)(3) for E_1 and E_2 the stability conditions become explicitly

$$\begin{gathered} \tau(i^2 + \varepsilon_t \varepsilon_r) > 0 \\ (B_{vv} - B_{uu})\tau\varepsilon_r + [\tau\varepsilon_t - (B_{vv} - B_{uu})](B_{uu} + \mu) > 0 \end{gathered} \tag{3.2)(6}$$

The first condition is always satisfied in practice. In order to simplify the second condition it is convenient to use the concept of <u>center of pressure</u> (C.P.) This is defined as the point of application of the total force on the body under a fixed angle of yaw. Denoting by $\varepsilon_p \ell$ its distance to the origin (dynamic center D.C. in this case) taken positive aft of this point Figure (3.2)(1) we have the relation,

$$Y\varepsilon_p \ell + N = 0$$

with

$$Y = -\rho U \frac{V}{\ell} A_{vv} v$$

$$N = \rho U V A_{rv} v$$

hence

$$A_{rv} = \varepsilon_p A_{vv} \tag{3.2)(7}$$

or

$$\tau\varepsilon_t - (B_{vv} - B_{uu}) = \tau\varepsilon_p \tag{3.2)(8}$$

Introducing this last value in (3.2)(3) the value of E_2 becomes

$$E_2 = \tau\left[(B_{vv} - B_{uu})\varepsilon_r + (B_{uu} + \mu)\varepsilon_p\right] \tag{3.2)(9}$$

and the second stability condition (3.2)(6)

$$(B_{vv} - B_{uu})\varepsilon_r + (B_{uu} + \mu)\varepsilon_p > 0 \tag{3.2)(10}$$

Since for elongated bodies $\mathcal{B}_{vv}-\mathcal{B}_{uu}$ is near unity and $\mathcal{B}_{uu}$ is small it is possible to write the approximate stability conditions

$$-\varepsilon_p < \frac{\varepsilon_r}{\mu} \qquad (3.2)(11)$$

In this form the main parameters are [figure (3.2)(1)]

μ the density of the projectile relative to the fluid

ε_p the distance of the center of pressure (C.P.) to the dynamic center (D.C.)

ε_r The distance of the rear point (R.P.) to the dynamic center (D.C.)

The significance of the approximate stability condition (3.2)(11) is illustrated geometrically by considering a point S located at a distance $\frac{\varepsilon_r \ell}{\mu}$ ahead of the dynamic center as shown in figure (3.2)(2). The stability condition is then that the center of pressure lies aft of S. It is seen that for bodies of low densities relative to the fluid the location of S and therefore also the stability is greatly dependent on the density.

Considering for instance the case $\mu = 1$ where the density of the body is the same as that of the surrounding fluid. This applies to airships, submarines or torpedoes. The dynamic center and the center of buoyancy will usually coincide very closely and lie around the center of the body. Assuming the R.P. to be near the fins, point S will be located near the nose.

Stability in this case will be obtained if the center of pressure lies somewhat behind the nose. This condition is relatively easy to satisfy and explains why only a small amount of fin area is used in airships, submarines and torpedoes.

Consider on the other hand a body of large relative density such as an aerial bomb. The value of μ being very large point S and the dynamic center coincide practically with the center of gravity. In such a case stability is obtained only by the well known condition that the center of pressure lies aft of the C.G. This condition is not so easily satisfied and usually requires a large amount of fin area.

3.3 Amount of stability of various bombs in air and water

Assuming the dynamic stability conditions to be satisfied it is important to know how stable the projectile is when submitted to a given perturbation. More specifically we may ask the following questions.

1) How quickly does the projectile return to a straight path?
2) Does the projectile oscillate or not during the transition state?

It is convenient to use the variable $s = \frac{Ut}{\ell}$ and the equations of motion in the form (3.1)(18) with the non dimensional unknowns $v^* = \frac{v}{U}$ $r^* = \frac{r\ell}{U}$,

$$\mu_y \frac{dv^*}{ds} = -C_{vv} v^* + C_{vr} r^*$$
$$\mu_y i^2 \frac{dr^*}{ds} = C_{rv} v^* - C_{rr} r^* \qquad (3.3)(1)$$

Note that the variable v^* represents the angle between the axis of the body and the velocity of the dynamic center. Also $r\ell$ is the difference in transversal velocities at both ends of the projectile.

The above questions are answered by considering again the equations obtained after the elimination of v^* between the two equations (3.3)(1). This equation already used in section (3.2) and numbered (3.2)(2) is,

$$\mu_y^2 i^2 \frac{d^2 r^*}{ds^2} + \mu_y E_1 \frac{dr^*}{ds} + E_2 r^* = 0 \qquad (3.3)(2)$$

with

$$E_1 = C_{vv} i^2 + C_{rr} = \tau (i^2 + \varepsilon_t \varepsilon_r)$$
$$E_2 = C_{vv} C_{rr} - C_{rv} C_{vr} = \tau \left[(B_{vv} - B_{uu}) \varepsilon_r + (B_{uu} + \mu) \varepsilon_p \right] \qquad (3.3)(3)$$

For the value of E_2 the expression (3.2)(9) is here used.

The general solution of equation (3.3)(2) is

$$r^* = C_1 e^{\lambda_1 s} + C_2 e^{\lambda_2 s} \qquad (3.3)(4)$$

with

$$\lambda_1 = \frac{1}{2\mu_y i^2}\left[-E_1 + \sqrt{E_1^2 - 4i^2 E_2}\right]$$
$$\lambda_2 = \frac{1}{2\mu_y i^2}\left[-E_1 - \sqrt{E_1^2 - 4i^2 E_2}\right] \qquad (3.3)(5)$$

C_1 and C_2 are arbitrary constants

For oscillations to occur the radical in the values of λ_1 and λ_2 must be imaginary. This condition is

$$E_1^2 - 4i^2 E_2 < 0$$

or

$$\tau(i^2 + \varepsilon_t \varepsilon_x)^2 - 4i^2\left[(B_{vv} - B_{uu})\varepsilon_x + (B_{uu} + \mu)\varepsilon_p\right] < 0 \qquad (3.3)(6)$$

It is seen that this condition is always satisfied if μ is large and if $\varepsilon_p > 0$. This is the case for a stable aerial bomb. A perturbation of the motion of such a bomb in the air results in a damped oscillation about the equilibrium direction. In order to derive single expressions for the wave length and the damping of this oscillation introduce the following approximation which holds for sufficiently large values of μ

$$\mu_y \cong \mu$$
$$E_1^2 - 4i^2 E_2 \cong -4i^2 \tau \mu \varepsilon_p \qquad (3.3)(7)$$

The general solution of (3.3)(2) may then be written

$$\tau^* = C_3 e^{-\frac{E_1}{2\mu i^2}s} \cos\left(\frac{1}{i}\sqrt{\frac{\tau\varepsilon_p}{\mu}}\, s + \varphi\right) \qquad (3.3)(8)$$

with C_3 and φ arbitrary constants. The ratio of the wavelength to the length l of the projectile is

$$s_1 = 2\pi i\sqrt{\frac{\mu}{\tau\varepsilon_p}} \qquad (3.3)(9)$$

The amplitude of the oscillation is decreased by a factor

$$e^{-\frac{E_1 \pi}{i\sqrt{\tau\varepsilon_p \mu}}} \qquad (3.3)(10)$$

when the projectile has travelled through a wavelength. The important parameters are here μ and ε_p

A number of numerical examples will now be considered. In the numerical applications it is useful to remember that the density of water is about 775 times that of air at sea level

Example I [figure (3.3)(1)]

Aerial bomb moving in air at sea level (assuming the density to be 3 relative to water)

The density relative to air is $\mu = 2320$

$\tau = 3$ $\qquad\qquad$ $\varepsilon_p = .10$

$\varepsilon_t = .4$ $\qquad\qquad$ $\varepsilon_n = .6$

$i = 1/5$

We find $s_1 = 2\pi i\sqrt{\dfrac{\mu}{\tau\,\varepsilon_p}} = 110$

The wavelength of the oscillation is 110 times the length of the projectile.

The amplitude of the oscillation is multiplied by the factor $e^{-\frac{\varepsilon_t \pi}{i\sqrt{\tau\varepsilon_p\mu}}} = e^{-.505} = .60$ when the projectile has travelled through a wavelength.

Example II [figure (3.3)(1)]

The same bomb as in example I moving in water

The density relative to water is $\mu = 3$

$\tau = 3$ $\qquad\qquad$ $\varepsilon_p = 0$

$\varepsilon_t = .3$ $\qquad\qquad$ $\varepsilon_n = .5$

$i = .23$

The value of i is slightly increased due to the contribution of the apparent mass to the moment of inertia. Due to the apparent mass of the tail the dynamic center is now 0.1ℓ aft of the C.G. This decreases by 0.1 the values of ε_t, ε_p, and ε_n. The value

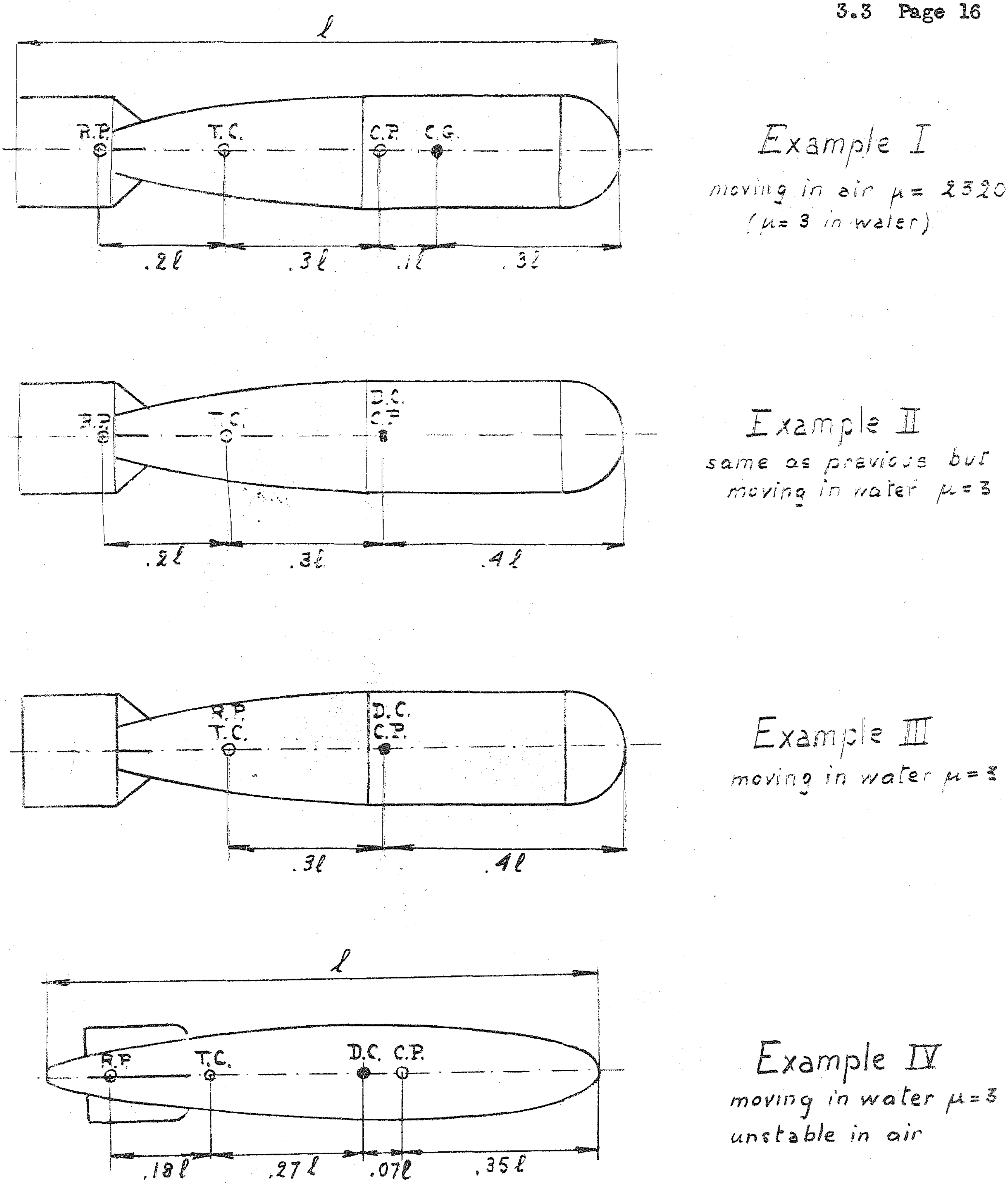

l
R.P.
T.C.
C.P.
C.G.
.2l
.3l
.1l
.3l
Example I
moving in air μ= 2320
(μ= 3 in water)
D.C.
C.P.
.2l
.3l
.4l
Example II
same as previous but
moving in water μ=3
R.P.
T.C.
D.C.
C.P.
.3l
.4l
Example III
moving in water μ=3
l
R.P.
T.C.
D.C.
C.P.
.18l
.27l
.07l
.35l
Example IV
moving in water μ=3
unstable in air

Figure (3.3)(1)

of τ is independent of the density and remains unchanged. Values of $\mathcal{B}_{vv}$ and $\mathcal{B}_{uu}$ and μ_y must also be introduced in this case. We assume

$$\mathcal{B}_{vv} - \mathcal{B}_{uu} = 1.1 \qquad \mathcal{B}_{uu} = .1$$

$$\mu_y = \mu + \mathcal{B}_{vv} = 4.2$$

The following values are derived

$$E_1 = .60 \qquad E_2 = 1.65$$

$$\lambda_1 = -1.56 \qquad \lambda_2 = -1.12$$

The angular velocity is given by

$$r^* = C_1 e^{-1.56s} + C_2 e^{-1.12s}$$

The motion in this case is non oscillatory and highly damped. The amplitude of the $e^{-1.12s}$ component of the motion is reduced to 32% of its original value after the projectile has travelled through a distance equal to its own length.

Example III [figure (3.3)(1)]

In order to evaluate the effect of an error in estimating the position of rear point consider again example II assuming the R.P. to be at the T.C. $(\varepsilon_t = \varepsilon_n = .3)$

We derive

$$E_1 = .429 \qquad E_2 = .99$$

$$\lambda_1 = -.96 + .36\sqrt{-1} \qquad \lambda_2 = -.96 - .36\sqrt{-1}$$

The angular velocity is given by

$$r^* = C_3 e^{-.96s} \cos(.36s + \varphi)$$

The motion this time is slightly oscillatory but the damping is still of the same order as in example II. The amplitude being reduced to 38% of its original value after the projectile has travelled through a distance equal to its own length.

Example IV [figure (3.3)(1)]

Consider a projectile shaped as the Goodyear Zeppelin incroduced in Chapter II section (2.6) and moving in water. Assume the dynamic center and the C.G. to coincide and be located at the center of buoyancy. The center of pressure is forward of the C.G. so that this projectile would be unstable in air.

The relative density is $\mu = 3$

$\tau = 2.48$	$\varepsilon_p = -.07$
$\varepsilon_t = .27$	$\varepsilon_2 = .45$
$i = .23$	$B_{vv} - B_{uu} = .9$
$\mu_y = 4$	$B_{uu} = .1$

We derive

$E_1 = .433$	$E_2 = .467$
$\lambda_1 = -2.06$	$\lambda_2 = -.318$

The angular velocity is given by

$$r^* = C_1 e^{-2.06 s} + C_2 e^{-.318 s}$$

The amplitude of the $e^{-.318 s}$ component of the motion is reduced to 38% of its original value after the projectile has travelled through a distance equal to three times its own length. The damping in this case is three times as small as in examples II and III.

3.4 General expressions for trajectory deviations caused by perturbations

We use the equations of motion (3.1)(18)

$$\mu_y \frac{dv^*}{ds} = -C_{vv} v^* + C_{vr} r^* \qquad (3.4)(1)$$
$$\mu_y i^2 \frac{dr^*}{ds} = C_{rv} v^* - C_{rr} r^*$$

and consider only plane motion. Small deviation from the straight path are only considered here. In this case three dimensional trajectories are obtained by superposition of plane motions in two perpendicular planes. What we are especially interested in is the angle between the initial trajectory and the new one caused by the perturbation. This angle will be referred to as the refraction.

Take the body axes x, y of the body to coincide with a fixed plane x', y' [figure (3.4)(1)] If the unknown valocities v^*, r^*, have been determined as functions of time by integrating equations (3.4)(1) the problem remains to find the trajectory. Assume that the trajectory is close to a fixed axis x' and that the angle α between the axis of symmetry (x) of the body and x' remains small. Them we may write approximately

$$x' = Ut$$
$$\frac{dy'}{dt} = \alpha U + v \qquad (3.4)(2)$$
$$\frac{d\alpha}{dt} = r$$

The first equation expresses that the motion in the x' direction is not affected by the perturbation. The deviation is given essentially by the two last equations (3.4)(2) changing to the variables r^*, v^*, $s = \frac{Ut}{l}$, $\eta = y'/l$ they become

$$\frac{d\eta}{ds} = \alpha + v^* \qquad (3.4)(3)$$
$$\frac{d\alpha}{ds} = r^*$$

Note that $\alpha + v^*$ is the angle of the velocity of the dynamic center with the fixed direction x', while v^* is the angle of this velocity with the

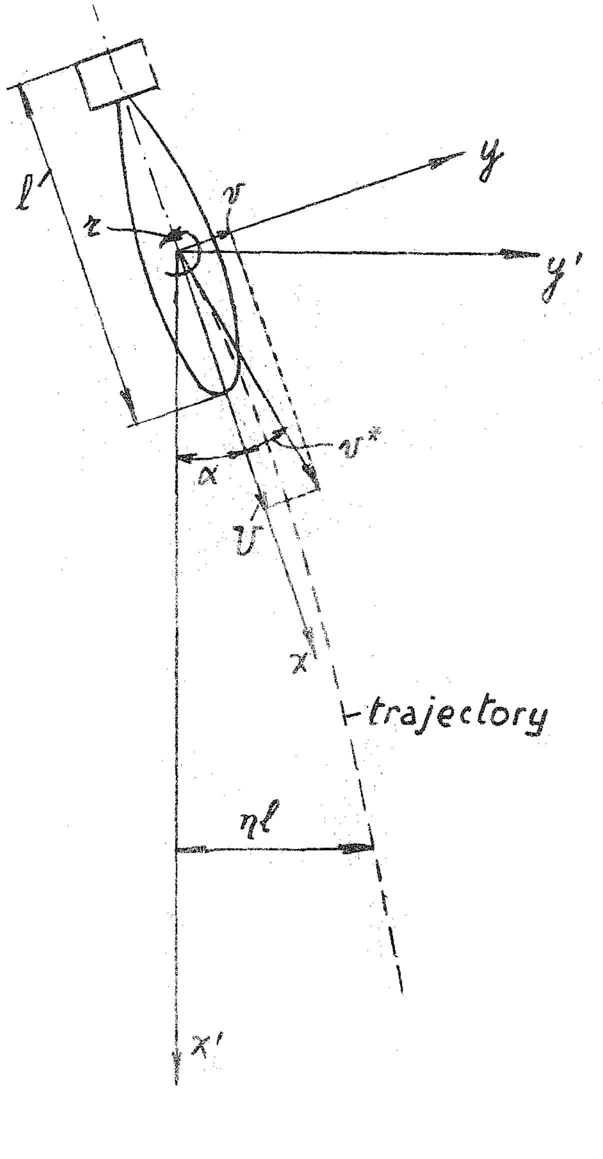


Figure (3.4) (1)

axis of the projectile.

The problem of finding the trajectory is that of integrating the system of four differential equations (3.4)(1) and (3.4)(3) under given initial conditions. The most convenient method of solution is the so called operational method. To apply this method write the equations in the form,

$$\begin{aligned} \mu_y \frac{dv^*}{ds} &= -C_{vv} v^* + C_{vr} r^* + F_v(s) \\ \mu_y i^2 \frac{dr^*}{ds} &= C_{rv} v^* - C_{rr} r^* + F_r(s) \\ \frac{d\alpha}{ds} &= r^* \\ \frac{d\eta}{ds} &= \alpha + v^* \end{aligned} \qquad (3.4)(4)$$

F_v and F_r are functions of time which might be taken to represent external forces so that the equations correspond to forced motion of the projectile. To use the operational method replace the operator $\frac{d}{ds}$ by the symbol λ,

$$\begin{aligned} \mu_y \lambda v^* &= -C_{vv} v^* + C_{vr} r^* + F_v(s) \\ \mu_y i^2 \lambda r^* &= C_{rv} v^* - C_{rr} r^* + F_r(s) \\ \lambda\alpha &= r^* \\ \lambda\eta &= \alpha + v^* \end{aligned} \qquad (3.4)(5)$$

and solve these equations algebraically for η

From the first two equations

$$\begin{aligned} v^* &= \frac{F_v}{\Delta(\lambda)}(\mu_y i^2 \lambda + C_{rr}) + \frac{F_r}{\Delta(\lambda)} C_{vr} \\ r^* &= \frac{F_v}{\Delta(\lambda)} C_{rv} + \frac{F_r}{\Delta(\lambda)}(\mu_y \lambda + C_{vv}) \end{aligned} \qquad (3.4)(6)$$

with

$$\Delta(\lambda) = \begin{vmatrix} \mu_y \lambda + C_{vv} & -C_{vr} \\ -C_{rv} & \mu_y i^2 \lambda + C_{rr} \end{vmatrix} \qquad (3.4)(7)$$

From the last two equations,

$$\eta = \frac{r^*}{\lambda^2} + \frac{v^*}{\lambda} \qquad (3.4)(8)$$

hence

$$\eta = \frac{F_v}{\Delta(\lambda)}\left(\frac{C_{rv}}{\lambda^2} + \mu_y i^2 + \frac{C_{rr}}{\lambda}\right) + \frac{F_r}{\lambda\Delta(\lambda)}\left(\mu_y + \frac{C_{vv}}{\lambda} + C_{vr}\right) \qquad (3.4)(9)$$

In order to make the functions F_v and F_r correspond to arbitrary initial values of v_o^* and r_o^* for $s = 0$ we introduce the unit impulse function $S(s)$. This function is zero everywhere except at $s = 0$ when it is infinite but in such a way that the area under the curve is finite and equal to unity. (reference 27 page 397). In other words

$$S(s) = \frac{d}{ds} 1(s) \qquad (3.4)(10)$$

where $1(s) = 0$ for $s < 0$ $1(s) = 1$ for $s > 0$. The function $1(s)$ is called the unit step function. Put

$$F_v = \mu_y v_o^* S(s) \qquad F_r = \mu_y i^2 r_o^* S(s) \qquad (3.4)(11)$$

It is easily seen by integrating once the two first equations No. (3.4)(5) that this amounts to introducing the arbitrary initial values v_o^* and r_o^*. Physically it is equivalent to giving the body a sudden lateral and rotational impulse by means of an instantaneous infinite force and couple.

The value of η becomes

$$\eta = \frac{\mu_y v_o^*}{\Delta(\lambda)}\left[\frac{C_{rv}}{\lambda^2} + \mu_y i^2 + \frac{C_{rr}}{\lambda}\right]S(s) + \frac{\mu_y i^2 r_o^*}{\lambda\Delta(\lambda)}\left[\mu_y + \frac{C_{vv}}{\lambda} + C_{vr}\right]S(s)$$

This is the operational expression for η as a function of s with the initial conditions $r^* = r_o^*$, $v = v_o^*$ and $\alpha = \eta = 0$ at $s = 0$. It is found convenient to consider also the case where the initial value of α is different from zero. The solution for this case is easily found by adding $\alpha_o s$ to the previous solution where α_o is the value of α for $s = 0$. We find

$$\eta = \frac{\mu_y v_o^*}{\Delta(\lambda)}\left[\frac{C_{rv}}{\lambda^2} + \mu_y i^2 + \frac{C_{rr}}{\lambda}\right]S(s)$$
$$+ \frac{\mu_y i^2 r_o^*}{\lambda\Delta(\lambda)}\left[\mu_y + \frac{C_{vv}}{\lambda} + C_{vr}\right]S(s) \qquad (3.4)(12)$$
$$+ \alpha_o s$$

This expression corresponds to initital values $r^* = r_o^*$

$v^* = v_o^*$ $\alpha = \alpha_o$ and $\eta = 0$ for $s = 0$

Consider the first case where the initial condition is an angular velocity with no yaw ($v_o^* = 0$, $\alpha_o = 0$) as shown in figure (3.4)(2).

$$\eta = \frac{\mu_y i^2 r_o^*}{\lambda\Delta(\lambda)}\left[\mu_y + \frac{C_{vv}}{\lambda} + C_{vr}\right]S(s) \qquad (3.4)(13)$$

This formula contains operational expressions of the form $\frac{1}{\lambda^2\Delta(\lambda)}$ and $\frac{1}{\lambda\Delta(\lambda)}$ which expanded in partial fractions are

$$\frac{1}{\lambda\Delta(\lambda)} = \frac{1}{\lambda\Delta(0)} + \frac{B_1}{\lambda-\lambda_1} + \frac{B_2}{\lambda-\lambda_2}$$
$$\frac{1}{\lambda^2\Delta(\lambda)} = \frac{1}{\lambda^2\Delta(0)} - \frac{\Delta'(0)}{\lambda\Delta^2(0)} + \frac{B_3}{\lambda-\lambda_1} + \frac{B_4}{\lambda-\lambda_2} \qquad (3.4)(14)$$

where the constants are,

λ_1 and λ_2 are the roots of the equation $\Delta(\lambda) = 0$

$$B_1 = \frac{1}{\lambda_1\Delta'(\lambda_1)} \qquad B_2 = \frac{1}{\lambda_2\Delta'(\lambda_2)}$$
$$B_3 = \frac{1}{\lambda_1^2\Delta'(\lambda_1)} \qquad B_4 = \frac{1}{\lambda_2^2\Delta'(\lambda_2)} \qquad (3.4)(15)$$
$$\Delta'(\lambda) = \frac{d}{d\lambda}\Delta(\lambda)$$

If the projectile is stable the terms containing the quantities $\lambda-\lambda_1$ and $\lambda-\lambda_2$ in the denominator correspond to damped motion and vanish at $s = \infty$. Therefore if we are interested only in the <u>asymptotic trajectory</u> at infinite time we need only retain the terms in $\frac{1}{\lambda}$ and $\frac{1}{\lambda^2}$. With this approximation,

$$\eta_\infty = \mu_y i^2 r_o^*\left[\frac{\mu_y + C_{vr}}{\Delta(0)}\frac{1}{\lambda} - C_{vv}\frac{\Delta'(0)}{\Delta^2(0)}\frac{1}{\lambda} + \frac{C_{vv}}{\lambda^2\Delta(0)}\right]S(s)$$

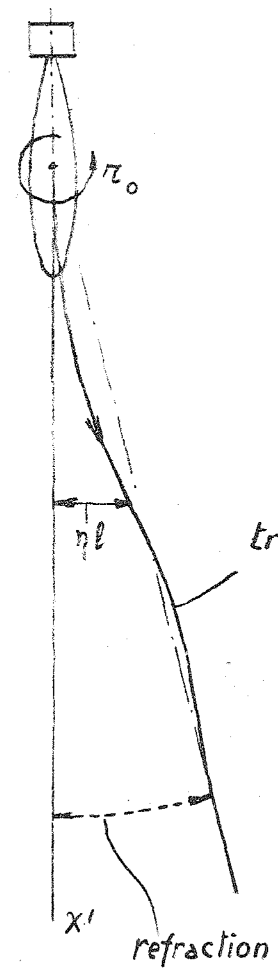


Figure (3.4)(2)

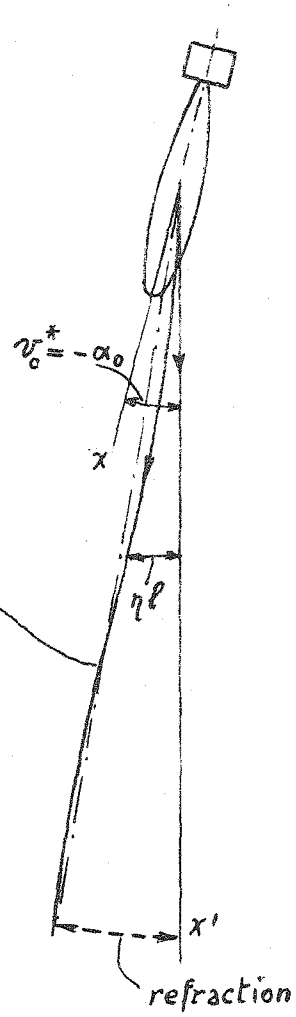


Figure (3.4)(3)

or $$\eta_\infty = i^2 r_0^* \left[\frac{b_1}{\lambda} + \frac{b_2}{\lambda^2} \right] S(s) \qquad (3.4)(16)$$

$$b_1 = \mu_y \frac{\mu_y + C_{vr}}{\Delta(0)} - \mu_y \frac{\Delta'(0)}{\Delta^2(0)} C_{vv}$$

$$b_2 = \mu_y \frac{C_{vv}}{\Delta(0)}$$

Now using the well known operational relations

$$\frac{1}{\lambda} S(s) = 1(s) \qquad \frac{1}{\lambda^2} S(s) = s\,1(s) \qquad (3.4)(17)$$

we obtain the asymptotic value of η_∞ as a function of s

$$\eta_\infty = i^2 r_0^* \left[b_1 + b_2 s \right] \qquad (3.4)(18)$$

This represents a straight line. The coefficient $i^2 r_0^* b_2$ is the angle between the original path and the new one after a sudden angular rotation r_0 has been imposed on the projectile under the assumption that there is no initial yaw ($v_0^* = \alpha_0 = 0$)

We now consider the case of an initial angle of yaw but not initial rotation ($r_0^* = 0$). It is convenient to assume the total initial velocity of the dynamic center to be along x'. This amounts to putting $v_0^* = -\alpha_0$ figure (3.4)(3) With these conditions we derive the asymptotic deviation by putting $r_0^* = 0$ and $\alpha_0 = -v_0^*$ in (3.4)(12). Proceeding as for the previous case we find,

$$\eta_\infty = v_0^* \left[b_3 + b_4 s \right] \qquad (3.4)(19)$$

with

$$b_3 = \mu_y \frac{C_{rr}}{\Delta(0)} - \mu_y \frac{\Delta'(0)}{\Delta^2(0)} C_{rv}$$

$$b_4 = \mu_y \frac{C_{rv}}{\Delta(0)} - 1 \qquad (3.4)(20)$$

It will be noted that this formula gives the deviation from the straight line along the direction of the initial velocity.

The expression $i^2 r_0^* b_2$ and $v_0^* b_4$ constitute what was referred to as the refraction of the trajectory.

3.5 Examples of deviations of trajectories in air and water under various initial perturbations.

In the previous section the following expressions are found for the deviation due to initial rotation $(r_0 = r_0^* \frac{V}{l})$ and angle of yaw $(v_0^* = \frac{v_0}{V})$ of the bomb,

$$\eta_\infty = i^2 r_0^*[b_1 + b_2 s]$$
$$\eta_\infty = v_0^*[b_3 + b_4 s] \qquad (3.5)(1)$$

We note that $\Delta'(0)$ and $\Delta(0)$ appearing in the coefficients b_1 b_2 b_3 b_4 have appeared already in sections (3.2) and (3.3) in connection with the symbols E_1 and E_2

$$\Delta'(0) = \mu_y E_1$$
$$\Delta(0) = E_2 \qquad (3.5)(2)$$

Hence we write

$$b_1 = \frac{\mu_y}{E_2}\left[\mu_y + C_{vr} - \frac{\mu_y E_1}{E_2} C_{vv}\right]$$
$$b_2 = \frac{\mu_y}{E_2} C_{vv}$$
$$b_3 = \frac{\mu_y}{E_2}\left[C_{rr} - \frac{\mu_y E_1}{E_2} C_{vv}\right] \qquad (3.5)(3)$$
$$b_4 = \frac{\mu_y}{E_2} C_{rv} - 1$$

Introducing the theoretical parameters from (3.1)(20), (3.2)(3) and (3.2)(9), we have

$$C_{vv} = \tau \qquad \mu_y + C_{vr} = \tau\varepsilon_r + B_{vv} - B_{uu}$$
$$C_{rv} = \tau\varepsilon_p \qquad C_{rr} = \tau\varepsilon_t\varepsilon_r$$
$$\mu_y E_1 = \mu_y \tau (i^2 + \varepsilon_t \varepsilon_r) \qquad (3.5)(4)$$
$$E_2 = \tau\left[(B_{vv} - B_{uu})\varepsilon_r + (B_{uu} + \mu)\varepsilon_p\right]$$

It is convenient to write simple expressions for the b's by introducing the theoretical values (3.5)(4) and using the approximation $B_{vv} = 1$, $B_{uu} = 0$,

$$b_1 = \frac{\mu_y(1+\tau\varepsilon_n)}{\tau(\varepsilon_n+\mu\varepsilon_p)} - \frac{\mu_y^2(i^2+\varepsilon_t\varepsilon_n)}{(\varepsilon_n+\mu\varepsilon_p)^2}$$

$$b_2 = \frac{\mu_y}{\varepsilon_n+\mu\varepsilon_p}$$

$$b_3 = \frac{\mu_y\varepsilon_t\varepsilon_n}{\varepsilon_n+\mu\varepsilon_p} - \frac{\mu_y^2(i^2+\varepsilon_t\varepsilon_n)\varepsilon_p}{(\varepsilon_n+\mu\varepsilon_p)^2} \qquad (3.5)(5)$$

$$b_4 = \frac{\mu_y\varepsilon_p}{\varepsilon_n+\mu\varepsilon_p} - 1$$

<u>Case of the aerial bomb</u>

It is interesting to note the values that these coefficients assume when the value of μ is very large as in the case of an aerial bomb. In this case

$$b_1 = \frac{1+\tau\varepsilon_n}{\tau\varepsilon_p} - \frac{i^2+\varepsilon_t\varepsilon_n}{\varepsilon_p^2}$$

$$b_2 = \frac{1}{\varepsilon_p}$$

$$b_3 = -\frac{i^2}{\varepsilon_p} \qquad (3.5)(6)$$

$$b_4 = 0$$

These values are independent of the density μ

For initial rotation the change of direction in the trajectory (refraction) is $i^2 r_0^* b_2 = \frac{i^2 r_0^*}{\varepsilon_p}$. It is inversely proportional to the distance between the center of pressure and the C.G. For an initial angle of yaw the change of direction is $v_0^* b_4 = 0$. The remarkable result is obtained that <u>for an aerial bomb there is no change of direction due to initial yaw</u> (refraction = 0)

<u>Numerical examples</u>

The same four examples are treated as in section (3.3). The formulae (3.5)(3) and (3.5)(4) are applied. The values of E_1 and E_2 were computed in (3.3)

<u>Example I</u> — Bomb in air. $\mu = 2320$ (See figure (3.3)(1))

$\tau = 3$ $\qquad \varepsilon_p = .10$ $\qquad \mathcal{B}_{vv} = 1.2$

$\varepsilon_t = .4$ $\qquad \varepsilon_r = .6$ $\qquad \mathcal{B}_{uu} = .1$

$i = 1/5$

we have

$C_{vv} = 3$ $\qquad \mu_y + C_{vr} = 2.9$

$C_{rv} = .3$ $\qquad C_{rr} = .72$

$\mu_y E_1 = 1950$ $\qquad E_2 = 697$

The deviations are

$$\eta_{\infty} = i^2 r_o^* \left[-18, 4 + 10 s\right] = r_o^* \left[-.735 + .4 s\right]$$

$$\eta_{\infty} = 0$$

Assuming the initial angular velocity such that $r_o \ell = 3$ ft/sec (ℓ = length of bomb), $U = 300$ ft/sec. The angle of deviation due to rotation is .004 radians or .22 degrees. The second formula indicates that the angle of deviation due to initial yaw, is zero.

This is in accordance with the approximation (3.5)(6). Trajectory deviations are illustrated in figure (3.5)(1)

<u>Example II</u> Bomb in water $\mu = 3$ (See figure (3.3)(1))

$\tau = 3$ $\qquad \varepsilon_p = 0$ $\qquad \mathcal{B}_{vv} = 1.2$

$\varepsilon_t = .3$ $\qquad \varepsilon_r = .5$ $\qquad \mathcal{B}_{uu} = .1$

$i = .23$ $\qquad \mu_y = 4.2$

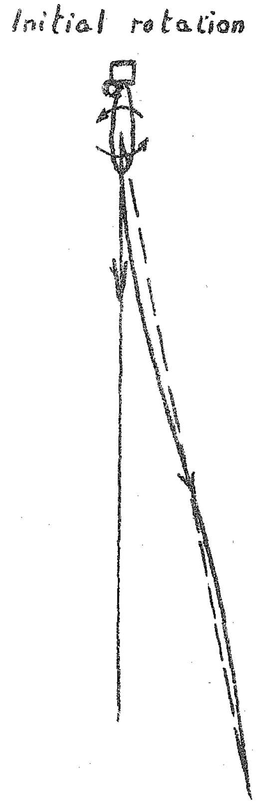
We have

$C_{vv} = 3$ $\qquad \mu_y + C_{vr} = 2.6$

$C_{rv} = 0$ $\qquad C_{rr} = .45$

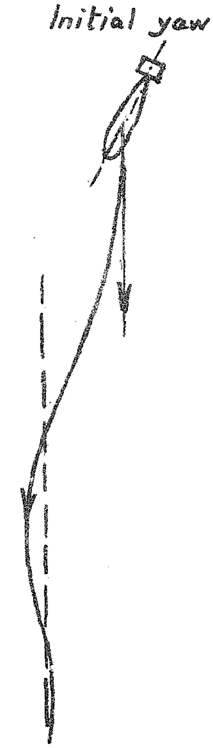
$\mu_y E_1 = 2.52$ $\qquad E_2 = 1.65$

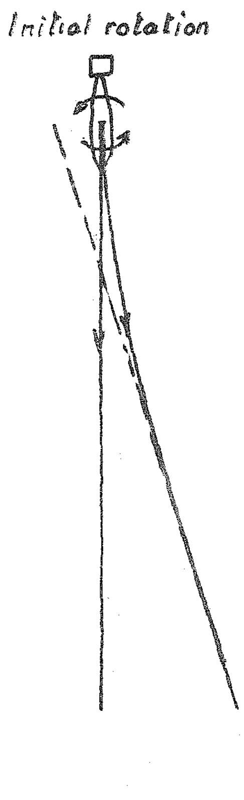
The deviations are



DEVIATION
IN AIR

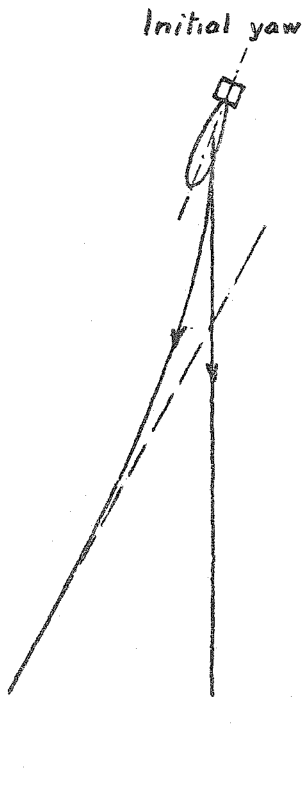
Figure (3.5)(1)





DEVIATION
IN WATER

Figure (3.5)(2)



$$\eta_{\infty} = i^2 r_0^* \left[- 5.05 + 7.65 s \right] = r_0^* \; (- .268 + .405 s)$$

$$\eta_{\infty} = v_0^* \;\; (1.15 - s)$$

Assuming the initial angular velocity such that $r_0 \ell = 3$ ft/sec. and $U = 30$ ft/sec the refraction is .0405 radians or 2.2 degrees.

The second formula indicates that initial yaw produces a refraction equal to the angle of yaw. The type of trajectories is shown in figure (3.5)(2)

<u>Example III</u> _ Bomb in water , $\mu = 3$ (See figure (3.3)(1))

Same as II except $\varepsilon_r = .3$

We have

$C_{vv} = 3$	$\mu_y + C_{vr} = 2.0$
$C_{rv} = 0$	$C_{rr} = .27$
$\mu_y E_1 = 1.80$	$E_2 = .99$

The deviations are

$$\eta = i^2 r_0^* \;\; (- 14.2 + 12.7 s) = r_0^* \;\; (- .75 + .67 s)$$

$$\eta = v_0^* \;\; (1.15 - s)$$

Assuming the initial angular velocity such that $r_0 \ell = 3$ ft/sec and $U = 30$ ft/sec the angle of deviation is .067 radians or 3.8 degrees. Initial yaw produces a refraction equal to the angle of yaw. The type of trajectories is illustrated in figure (3.5)(2).

<u>Example IV</u> _ Bomb in water $\mu = 3$ _ (unstable in air - stable in water) (see figure (3.3)(1))

$\tau = 2.48$	$\varepsilon_p = -.07$	$B_{vv} = 1$
$\varepsilon_t = .27$	$\varepsilon_r = .45$	$B_{uu} = .1$
$i = .23$	$\mu_y = 4$	

We have

$C_{vv} = 2.48$ $\qquad \mu_y + C_{rr} = 2.02$

$C_{rv} = -.174$ $\qquad C_{rr} = .302$

$\mu_y E_1 = 1.73$ $\qquad E_2 = .467$

The deviations are

$$\eta = i^2 r_o^* \, (-61.5 + 21.2 s) = r_o^* \, (-3.26 + 1.12 s)$$

$$\eta = v_o^* \, (8.08 - 2.5 s)$$

Assuming $r_o \ell = 3$ ft/sec and $U = 30$ ft/sec the angle of deviation is .112 radians or 6.3 degrees. The second formula indicates that an initial yaw angle produces an angular deviation of the trajectory equal to 2-1/2 times the angle of yaw.

The type of trajectories is illustrated in figure (3.5)(2). Note that the refraction is larger than the initial yaw. This is always the case when the C.P. lies in front of the D.C.

Conclusions

The equations of motion of a projectile in a fluid are simplified by the introduction of a point referred to as the dynamic center. This point generalizes the concept of center of gravity for a body moving in a dense fluid by taking into account the mass of the surrounding fluid.

The stability of rectilinear motion is defined by the condition that if a perturbation is imposed the projectile will go back to rectilinear motion. The direction of the new trajectory being usually different from the initial one.

An approximate simplified stability criterion is derived. It involves the relative density of body and fluid and the relative distances between the rear point (R.P.), the dynamic center (D.C.), and the center of pressure (C.P.). As far as it is justified to discard Reynolds Number effects on the location of R.P. and C.P. the stability condition is independent of the velocity. It is found that for aerial bombs stability requires the C.P. to lie behind the C.G. (identical with D.C. in this case). For bodies moving in water the C.P. may very well lie ahead of the D.C. For the extreme case where the density of the body is equal to that of the fluid stability is still obtained if the C.P. lies somewhat behind the nose. This explains why bodies such as airships, submarines, and torpedoes require a relatively small amount of fin area compared with aerial bombs.

The amount of stability after a perturbation depends on how quickly the projectile returns to a rectilinear trajectory. It is found that a perturbation on an aerial bomb causes a damped oscillation which depends mainly on the density and the distance of the C.P. to the C.G. The wave length of the oscillation in the example (I) investigated is of the order of hundred times the length of the bomb. In water oscillations do not generally occur or are nearly critically damped; the transient motion is them of the nature of a subsidence. Investigation of three projectiles (II, III, IV) of density three relative to water reveals that when moving in water and submitted to a perturbation they return very quickly to a straight path after travelling through a distance of about two to five times their own length. Example IV considers a body unstable in air but it is found to be highly stable in water.

When comparing stability in air and water one of the differences if of course the position of the dynamic center. In water the fins constitute a transversal apparent mass which will move the D.C. toward the rear. This would tend to decrease the stability but the theory shows that this factor is not the most important and that generally when no cavitation is present and gravity is neglected the stability of a bomb will be much greater in water than in air.

When a perturbation is imposed on the projectile such as a sudden velocity of rotation or a sudden angle of yaw a change in the trajectory occurs. After a certain period of transition during which oscillations of the projectile may or may not take place the projectile if stable will move again on a rectilinear trajectory. The new trajectory however will generally lie at an angle with the initial one. This angle is referred to as the refraction and the new rectilinear path as the asymptotic trajectory. It was shown by the use of the operational calculus that it is possible to find general expressions for the asymptotic trajectory and the refraction directly in terms of the projectile parameters and initial conditions without going through the process of solving the differential equations of motion.

Application of the formulae show that in aerial bombs the refraction due to initial yaw is zero. In water refraction due to yaw becomes greater than the initial yaw if the projectile is unstable in air. The refraction due to initial velocity of rotation is found to be small in the examples treated.

Attention must be called to the fact that a large amount of stability is not always advisable as this makes the motion of the body more sensitive to disturbances in the fluid. For instance in the case of torpedoes great stability would make the body very sensitive to wave disturbances while a very stable airship would be too sensitive to gusts and loose maneuverability.

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September 1 - 1942

List of Symbols

x, y, z	coordinates along body axes (moving with the body)
u, v, w	velocity components of the origin parallel to the instantaneous position of x, y, z.
p, q, r	components of angular velocity parallel to the instantaneous position of x, y, z.
T	kinetic energy of the fluid due to the motion of the body
A	longitudinal apparent mass of the fluid (x direction)
B	transversal apparent mass of the fluid (y direction)
R	rotational apparent mass of the fluid about OZ
A' P Q G'' etc.	coefficients in expression (1.1)(1).
X, Y, Z	components of the force exerted on the body by the fluid parallel to x, y, z.
L, M, N	moments of the forces exerted on the body by the fluid about x, y, z.
$\dot{u}, \dot{v}$, etc.	derivatives of u, v, etc. with respect to time.
U	large velocity component of the body in the x direction
α	angle of yaw (between x and x')
x' y'	axes with fixed directions.
X' Y'	force components of the fluid on the body parallel to the fixed directions x', y'.
u' v'	velocity components of the origin parallel to the fixed directions x', y'.
x_1	distance of virtual center of mass to the origin O.
v_1	velocity of the virtual center of mass parallel to y.
R_1	rotational apparent mass about the virtual center of mass.
k_1 k_2	inertia coefficient of the prolate ellipsoid for longitudinal and transversal apparent mass.
k'	inertia coefficient for the rotational apparent mass of the ellipsoid.
I_f	moment of inertia of virtual volume of fluid and moment of inertia of displaced fluid (in Chapter I)

V	volume of displaced fluid
ρ	mass of fluid per unit volume
ℓ	length of projectile
B_{vv} B_{uu} ... etc.	inertia coefficients (acceleration derivatives defined by (2.1)(1)
A_{vv} A_{uu} ... etc.	velocity derivatives defined by (2.1)(1)
D	drag
ω	circular frequency of oscillation
X" Y" N"	force components and moment of the fluid on the body when the origin is at O"
u" v" r"	velocity components when the origin is at O"
B''_{uu} B''_{vv} .... A''_{uu} A''_{vv}	stability derivatives when the origin is at O"
$\varepsilon\ell$	distance of O to O"
I_z	moment of inertia of the body about a transversal axis
k	spring constant
Y_t	transversal force due to tail action
$\ell\varepsilon_c$, $\ell\varepsilon_r$, $\ell\varepsilon_t$,	distance of the origin to virtual center of mass (V.C.) rear point (R.P.) and tail center (T.C.) Figure (2.5)(3)
v_c	transversal velocity at V.C.
v_r	transversal velocity at R.P.
τ	tail lift factor
C_L C_D C_m	lift, drag, and moment coefficients for airship models.
v_g	transversal (y) component of the velocity of the C. G.
$\bar{I}_b$	moment of inertia of body about a transversal axis through the C.G.
a	distance of the C.G. of the body to the origin of the body axes.
m	mass of body
s	$= \frac{Ut}{\ell}$

t	time
$\frac{m}{\rho V} = \mu$	density of body relative to the fluid
$k = \sqrt{I_b/m}$	radius of gyration of the body about the dynamic center
$\mu_y = \mu + B_{vv}$	transversal density
$\mu_y i^2 = \mu \frac{k^2}{\ell^2} + B_{rr}$	
i	a radius of gyration expressed as a fraction of the length for the total transversal mass (body and fluid) about the dynamic center
$C_{vv} = A_{vv}$	
$C_{vr} = A_{vr} - \mu$	
$C_{rv} = A_{rv}$	
$C_{rr} = A_{rr} + \varepsilon_c B_{vv}$	
$v^* = \frac{v}{U}$ = yaw	
$r^* = \frac{r\ell}{U}$	
$E_1 = C_{vv} i^2 + C_{rr}$	
$E_2 = C_{vv} C_{rr} - C_{rv} C_{vr}$	
λ_1 λ_2	characteristic exponents in the general solution of the equations of motion
$\ell\varepsilon_p$	distance of the center of pressure to the dynamic center (positive when aft of the D.C.)
s_1	wave length of oscillation divided by ℓ
$F_v(s)$ $F_r(s)$	arbitrary external forces acting on the body
$\eta = \frac{y'}{\ell}$	distance of the D.C. from the fixed axis x' measured as a multiple of the length ℓ
$\lambda = \frac{d}{ds}$	differential operator
$\Delta(\lambda)$	determinant (3.4)(7) $\quad \Delta'(\lambda) = \frac{d}{d\lambda}\Delta(\lambda)$
$\Delta(0)$	value of $\Delta(\lambda)$ for $\lambda = 0$
$S(s)$	unit impulse function $\quad \frac{d}{ds} 1(s) = S(s)$
$1(s)$	unit step function $\quad 1(s) = 0$ for $s < 0$; $1(s) = 1$ for $s > 0$

r_0 initial angular velocity about a transversal axis through the D.C. (for $\sigma = 0$)

v_0 initial transversal velocity of the D.C. for $\sigma = 0$

$r_0^* = \dfrac{r_0 l}{U}$

$v_0^* = \dfrac{v_0}{U}$ initial yaw for $\sigma = 0$

a_0 initial angle between the body axis x and the fixed direction X'

B_1, B_2, B_3, B_4 see (3.4)(15)

b_1, b_2, b_3, b_4 coefficients in the expressions (3.4)(18) and (3.4)(19) for the asymptotic disturbed trajectories

η_∞ value of η for asymptotic trajectory

DYNAMIC STABILITY OF BOMBS

AND PROJECTILES

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Chapter IV

Stability of the Vertical Fall

Introduction

In the previous chapter the dynamic stability was investigated for what might be designated as "free motion", i.e. the motion when the effect of gravity is neglected.

The object of the present chapter is to analyze the additional effect of gravity on the stability in a vertical fall.

The equations for a trajectory are linearized by considering small deviations from the vertical. A further simplification is introduced by the assumption that the velocity of fall is constant. It is believed that the results still hold in a qualitative way for conditions which do not satisfy these assumptions very closely.

The applications of the general formulae are limited to subsonic velocities in air and to bodies moving in water without cavitation or air bubble. The stability criterion is applied to numerical examples for aerial and underwater bombs.

4.1 Equations of motion under gravity

The motion is referred to body axes x, y and the fixed axes x' y' located both in the plane of the motion (Figure (4.1) (1)). These are the same as in previous chapters with the specification that x' is a vertical axis. The angle α denotes the angle of the axis of the body with the vertical direction.

The equations are the same as those derived in section (3.1) of Chapter III except that the weight and the buoyancy must now be added. These contribute the following terms to the x and y components of the forces and the moment about the D.C.

$$\begin{aligned} X_g &= \rho V g(\mu-1)\cos\alpha \\ Y_g &= -\rho V g(\mu-1)\sin\alpha \\ N_g &= -\rho V g l(\varepsilon_g\mu+\varepsilon_b)\sin\alpha \end{aligned} \qquad (4.1)\ (1)$$

The equations of motion now read

$$\begin{aligned} m\dot{U} &= -D + X_g \\ m\dot{v} + ma\dot{r} + mUr &= Y + Y_g \\ ma\dot{v} + I_\ell \dot{r} + maUr &= N + N_g \end{aligned} \qquad (4.1)\ (2)$$

The symbol D denotes the drag associated with the velocity U, and Y, N are the hydrodynamical forces as expressed by (3.1) (7) Substituting these values in the equations of motion (4.1) (2) dividing the first and second equations by ρV and the third by $\rho V l$ and locating the origin at the D.C. they become

$$\begin{aligned} \mu\dot{U} &= -\frac{A_{uu}}{2l}U^2 + g(\mu-1)\cos\alpha \\ \mu\dot{v} &= \frac{U}{l}\left[-A_{vv}v + (A_{vr}-\mu)\dot{\alpha}l\right] - g(\mu-1)\sin\alpha \\ \mu_y i^2\ddot{\alpha}l &= \frac{U}{l}\left[A_{rv}v - (A_{rr}+\varepsilon_c B_{vv})\dot{\alpha}l\right] - g(\varepsilon_g\mu+\varepsilon_b)\sin\alpha \end{aligned} \qquad (4.1)\ (3)$$

Note that r has been replaced by $\dot{\alpha}$ and $\dot{r}$ by $\ddot{\alpha}$. These are the most general equations for the bomb trajectory under any arbitrary initial conditions. It will now be assumed that the velocity U and the density ρ are constant. Introducing the new variables

$$s = \frac{Ut}{\ell} \qquad v^* = \frac{v}{U}$$

and the constants

$$G_v = \frac{g\ell}{U^2}(\mu - 1) \tag{(4.1) (4)}$$

$$G_\alpha = \frac{g\ell}{U^2}(\varepsilon_g \mu + \varepsilon_b)$$

the last two equations (4.1) (3) become

$$\mu_y \frac{dv^*}{ds} = -C_{vv} v^* + C_{vr} \frac{d\alpha}{ds} - G_v \sin\alpha$$

$$\mu_y i^2 \frac{d^2\alpha}{ds^2} = C_{rv} v^* - C_{rr} \frac{d\alpha}{ds} - G_\alpha \sin\alpha \tag{(4.1) (5)}$$

Values of μ_y, $\mu_y i^2$, and the C coefficients have been defined in Chapter III. These equations are non-linear only through $\sin\alpha$. They become linear insofar as it is justified to replace $\sin\alpha$ by α i.e. if the angle between the axis of the projectile and the vertical is not too large. With this approximation

$$\mu_y \frac{dv^*}{ds} = -C_{vv} v^* + C_{vr} \frac{d\alpha}{ds} - G_v \alpha \tag{(4.1) (6)}$$

$$\mu_y i^2 \frac{d^2\alpha}{ds^2} = C_{rv} v^* - C_{rr} \frac{d\alpha}{ds} - G_\alpha \alpha$$

These are the basic equations to be used in the present analysis.

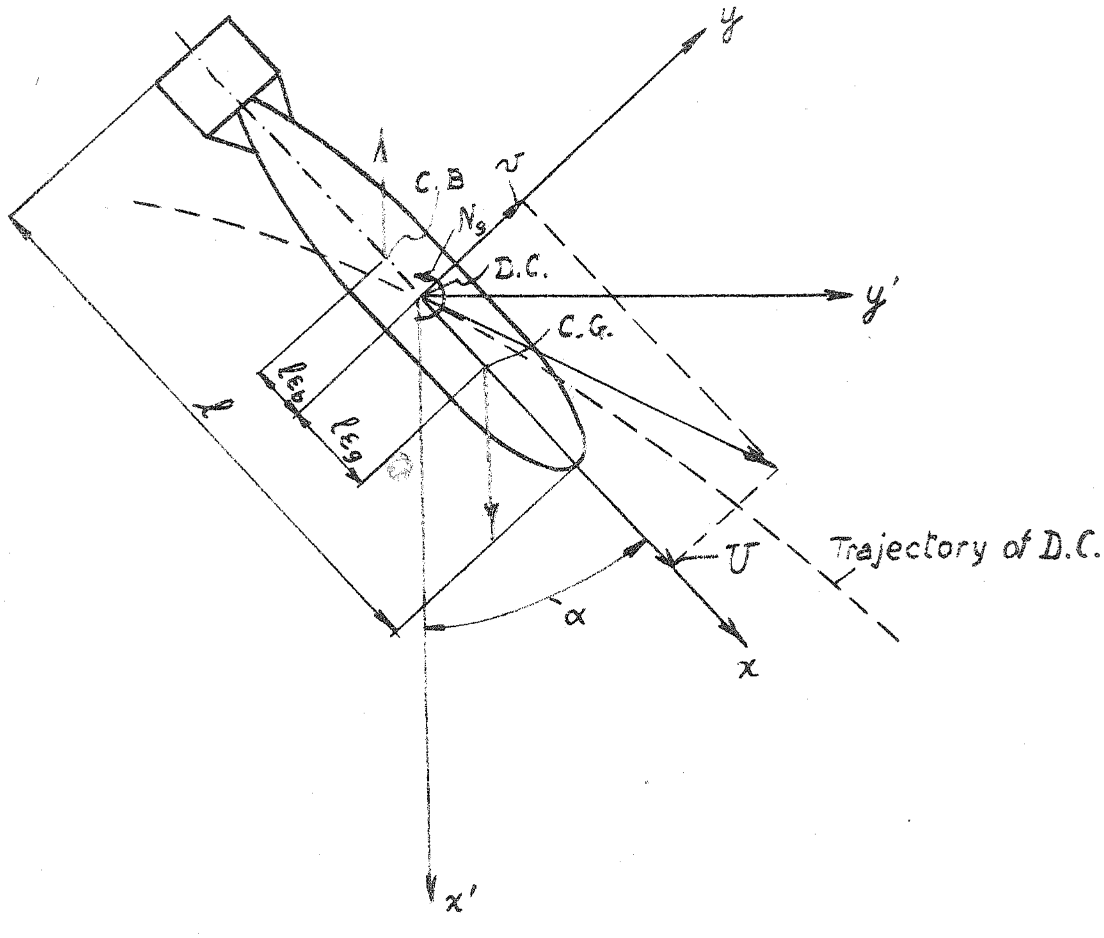

FIGURE (4.1)(1)

C.B. CENTER OF BUOYANCY
D.C. DYNAMIC CENTER
C.G. CENTER OF GRAVITY

4.2 Stability Criterion

The elimination of v* between the two equations (4.1) (6) leads to the following differential equation for α

$$\mu_y^2 i^2 \frac{d^3\alpha}{ds^3} + \mu_y (C_{zz} + C_{vv} i^2) \frac{d^2\alpha}{ds^2} +$$
$$(\mu_y G_\alpha + C_{vv} C_{zz} - C_{vz} C_{zv}) \frac{d\alpha}{ds} + (C_{vv} G_\alpha + C_{zv} G_v) = 0 \quad (4.2)\ (1)$$

Write the characteristic equation

$$A\lambda^3 + B\lambda^2 + C\lambda + D = 0$$
$$A = \mu_y^2 i^2 \quad (4.2)\ (2)$$
$$B = \mu_y (C_{zz} + C_{vv} i^2) = \mu_y E_1$$
$$C = \mu_y G_\alpha + C_{vv} C_{zz} - C_{vz} C_{zv} = \mu_y G_\alpha + E_2$$
$$D = C_{vv} G_\alpha + C_{zv} G_v$$

The quantities E_1 E_2 are the basic criteria defined in Chapter III for the dynamic stability of free motion. The general solution of (4.2) (1) reads

$$\alpha = C_1 e^{\lambda_1 s} + C_2 e^{\lambda_2 s} + C_3 e^{\lambda_3 s} \quad (4.2)\ (3)$$

where λ_1 λ_2 λ_3 are the roots of the characteristic equation (4.2) (2). Stability requires that if the roots λ are real they are all negative and if there are complex the real part must be negative. The general stability conditions for a cubic when A>0, are (reference 27)

$$B > 0 \quad C > 0 \quad D > 0$$
$$BC - AD > 0 \quad (4.2)\ (4)$$

Since A and B in the present case are assumed by nature to be positive it follows that C>0 is superfluous. Therefore the conditions reduce to

$$D > 0$$
$$BC - AD > 0 \quad (4.2)\ (5)$$

The significance of these conditions is discussed in the next section.

The assumption that $B > 0$, might not be legitimate in certain cases of cavitation, but we discard this case at the present stage of the analysis.

4.3 Discussion of Stability Conditions

It was found in the previous section that necessary and sufficient conditions for stability are

$$D > 0 \qquad (4.3)\ (1)$$

$$B\,C - A\,D > 0 \qquad (4.3)\ (2)$$

The first condition (4.3) (1) is

$$D = C_{vv} G_{\alpha} + C_{zv} G_{v} > 0 \qquad (4.3)\ (3)$$

From (4.1) (4), (3.1) (20) and (3.2) (8) it may be written

$$C_{vv}(\varepsilon_g \mu + \varepsilon_b) + C_{zv}(\mu - 1) > 0 \qquad (4.3)\ (4)$$

or

$$\varepsilon_g \mu + \varepsilon_b + \varepsilon_p(\mu - 1) > 0 \qquad (4.3)\ (5)$$

which is <u>independent of the velocity</u>. The sign of $\varepsilon_g \mu + \varepsilon_b$ is the same as that of the gravity righting moment. It is assumed here that $\mu > 1$ therefore, the criterion (4.3) (5) is always fulfilled if the gravity righting moment is positive and if the C.P. lies aft of the D.C. $\varepsilon_p > 0$. If the gravity moment is stabilizing ε_p may become negative, but if the gravity moment is destabilizing ε_p must necessarily be positive for stability.

When the condition (4.3) (3) is not fulfilled, i.e. when $D < 0$ the characteristic equation (4.2) (2) has a positive real root and the instability is of the nature of a divergence.

The second condition (4.3) (2) is

$$\mu_y E_1(\mu_y G_\alpha + E_2) > \mu_y^2 i^2 (C_{vv} G_\alpha + C_{zv} G_v) \qquad (4.3)\ (6)$$

By substituting the values of these coefficients and by cancellation of the term $\mu_y^2 i^2 C_{vv} G_\alpha$ this relation becomes

$$E_1 E_2 > \frac{g l \mu_y}{U^2}\left[i^2(\mu-1)C_{zv} - (\varepsilon_g \mu + \varepsilon_b)C_{zz}\right] \qquad (4.3)\ (7)$$

where E_1, E_2 are the two quantities controlling the dynamic stability of free motion. Let us assume that the free motion is stable, i.e. $E_1\ E_2 > 0$. Two cases must be distinguished.

(a)
$$i^2(\mu-1)C_{zv} - (\varepsilon_g \mu + \varepsilon_b)C_{zz} > 0 \qquad (4.3)\ (8)$$

in this case instability exists below a critical speed U_c given by the equation

$$E_1 E_2 = \frac{g l \mu_y}{U_c^2}\left[i^2(\mu-1)C_{zv} - (\varepsilon_g \mu + \varepsilon_b)C_{zz}\right] \qquad (4.3)\ (9)$$

Above this critical speed the criterion is satisfied.

(b)
$$i^2(\mu-1)C_{zv} - (\varepsilon_g \mu + \varepsilon_b)C_{zz} < 0 \qquad (4.3)\ (10)$$

In this case the criterion is satisfied at all speeds.

The significance of the present criterion is understood when we notice that if all the coefficients of the characteristic equation are positive it cannot have any positive real root. Therefore if the criterion (4.3) (6) is not satisfied it corresponds to the existence of an undamped oscillation.

For instance the limiting case

$$BC - AD = 0 \qquad (4.3)\ (11)$$

occurs at the critical velocity as given by (4.3) (9)

In this case it is easily verified that there exists a harmonic solution

$$\alpha = C_4 \cos(\nu s + \varphi) \qquad (4.3)\ (12)$$

with

$$\nu^2 = \frac{C}{A} = \frac{D}{B}$$

or

$$\nu^2 = \frac{C_{vv} G_\alpha + C_{rv} G_v}{\mu_y E_1} \tag{(4.3) (13)}$$

The period T of this oscillation is

$$T = \frac{2\pi l}{U_c \nu} = 2\pi \sqrt{\frac{l}{g}} \sqrt{\frac{\mu_y E_1}{C_{vv}(\varepsilon_g \mu + \varepsilon_b) + C_{rv}(\mu - 1)}} \tag{(4.3) (14)}$$

4.4 Stability of the Vertical Fall in Air

Application of the previous results will now be made to cases such as aerial bombs where the density of the projectile relative to the fluid is very high.

Consider the stability condition D> is equivalent to

$$\varepsilon_g + \frac{\varepsilon_b}{\mu} + \varepsilon_p\left(1 - \frac{1}{\mu}\right) > 0 \qquad (4.4)\ (1)$$

For large values of μ the C. G. tends to coincide with the dynamic center so that ε_g tends to zero. Therefore the condition D> 0 in the case of aerial bombs reduces to

$$\varepsilon_p > 0 \qquad (4.4)\ (2)$$

which is identical with the condition $E_2 > 0$ for dynamic stability of the free motion as found in Chapter III.

The second stability condition

$$BC - AD > 0 \qquad (4.4)\ (3)$$

or explicitly

$$E_1 E_2 > \frac{g\ell\mu_y}{U^2}\left[i^2(\mu-1)C_{zv} - (\varepsilon_g\mu + \varepsilon_b)\,C_{zz}\right] \qquad (4.4)\ (4)$$

is also simplified under the assumption that μ is large. It is possible to replace C_{vz} by $-\mu$ so that E_2 becomes μA_{zv}. Also μ_y is replaced by μ. The stability condition becomes

$$E_1 > \frac{g\ell\mu i^2}{U^2} \qquad (4.4)\ (5)$$

The critical velocity is

$$U_c = i\sqrt{\frac{g\ell\mu}{E_1}} \qquad (4.4)\ (6)$$

Below this velocity the vertical fall in the air is unstable and the instability appears in the form of undamped oscillations of the trajectory about the vertical.

Consider for instance the bomb shown in Example I of Figure (3.3) Chapter III, for which

$$\tau = 3 \qquad \varepsilon_p = .10$$
$$\varepsilon_t = .4 \qquad \varepsilon_z = .6$$
$$i = .2 \qquad \mu = 2320$$

(density 3 relative to water)

We have $E_1 = \tau(i^2 + \varepsilon_t \varepsilon_z) = .84$

Assuming the length of the bomb $\ell = 3$ ft., the formula (4.4) (6) gives for the critical velocity

$$V_c = 104 \text{ ft/sec.}$$

The velocity of release of the bomb will usually be much above this critical value except in the case of retrofiring. The period of oscillation at this velocity is given by (4.3) (14). Introducing the assumption of large density this expression becomes

$$T = 2\pi\sqrt{\frac{\ell}{g}}\sqrt{\frac{E_1}{A_{zv}}} \qquad (4.4)\ (7)$$

with $E_1 = \tau(i^2 + \varepsilon_t \varepsilon_z)$

$A_{zv} = \tau \varepsilon_p$

$$T = 2\pi\sqrt{\frac{\ell}{g}}\sqrt{\frac{i^2 + \varepsilon_t \varepsilon_z}{\varepsilon_p}} \qquad (4.4)\ (8)$$

In the present example we find

$$T = 3.2 \text{ sec.}$$

The wave length at the critical speed is

$$V_c T = 332 \text{ ft.}$$

Note that T is <u>independent of the density</u>.

It can be seen from the formula (4.4) (6) that the critical velocities depend to a large extent on the radius of gyration. A large value of i raises U_c thereby increasing the range of unstable velocities.

It is possible to find the condition under which the terminal velocity is equal to the critical. For large values of μ the terminal velocity is

$$U_t = \sqrt{\frac{2g\ell\mu}{A_{uu}}} \qquad (4.4)\ (9)$$

where A_{uu} is the drag coefficient defined above.
The total drag D may be written in terms of A_{uu} or by means of the drag coefficient c_D referred to the cross sectional area of the body of the bomb.

$$D = A_{uu}\,\rho\frac{U^2}{2}\frac{V}{\ell} = c_D\,\rho\frac{U^2}{2}S \qquad (4.4)\ (10)$$

We have the relation

$$A_{uu} = c_D\frac{S\ell}{V} \qquad (4.4)\ (11)$$

Putting $U_t = U_c$ we find

$$A_{uu} = \frac{2E_1}{i^2} \qquad (4.4)\ (12)$$

This is the value of the drag coefficient such that the terminal velocity in air is equal to the critical.

In the numerical example of this section the drag necessary to obtain a terminal velocity equal to the critical value, i.e. 104 ft/sec., is very high. However, this drag is extremely sensitive to the magnitude of the radius of gyration. For instance, by putting i = .5 i.e. by assuming all the mass of the bomb to be concentrated at both ends we find $E_1 = 1.47$ and

$$A_{uu} = 11.7$$

Assuming $\frac{Sl}{V} = .8$ this corresponds to a value of the drag coefficient

$$C_D = 9.4$$

Such a drag coefficient can be obtained by adding to the bomb a tail disc or flat cone of a diameter equal to approximately three times that of the bomb.

The results obtained here point to dynamic instability in the vertical fall of bombs with high drag and large radius of gyration. A typical example of violent instability in vertical fall is that of a disc of cardboard falling perpendicularly to its plane. Although this shape is stable in free motion because the relation between center of pressure and yaw is such that the cardboard always tends to move broadside on, it is a matter of common experience that the fall is not straight and that the path oscillates considerably about the vertical.

4.5 Stability of the Vertical Fall in Water

The condition D>0 or (4.3) (5) is

$$\varepsilon_g\mu + \varepsilon_b + \varepsilon_p(\mu - 1) > 0 \qquad (4.5)\ (1)$$

It is independent of the velocity. Since the vertical fall is here investigated it is assumed that $\mu > 1$. In that case the criterion is satisfied if $\varepsilon_g\mu + \varepsilon_b > 0$ i.e. if the buoyancy and gravity produce a righting moment about the D.C. and if $\varepsilon_p > 0$ i.e. if the C.P. lies aft of the D.C.

In order to investigate the criterion

$$BC - AD > 0 \qquad (4.5)\ (2)$$

we first have to consider the existence of a critical velocity, i.e. the sign of the expression

$$i^2(\mu - 1)\,C_{zv} - (\varepsilon_g\mu + \varepsilon_b)\,C_{zz} \qquad (4.5)\ (3)$$

Take the bomb shown in example II of Figure (3.3) (1) in Chapter III

$\tau = 3$	$\varepsilon_p = 0$
$\varepsilon_t = .3$	$\varepsilon_z = .5$
$\varepsilon_g = .15$	$\varepsilon_b = 0$
$i = .23$	$\mu = 3$
$\mu_y = 4.2$	

Then $C_{zv} = 0$ and the expression (4.5) (3) is negative. Therefore the bomb is stable at all velocities.

The motion of the bomb under an initial disturbance is determined by the roots of the characteristic equation (4.2) (2). The values of E_1 and E_2 are computed in Chapter III.

Assuming a length $\ell = 3$ ft. and a velocity of 25 ft/sec.,

$$G_v = .306 \qquad G_\alpha = .0685$$

the characteristic equation is

(4.5) (4)

The roots are

$$\lambda_1 = -.127 \qquad (4.5)\ (5)$$
$$\lambda_2 = -1.28 + .3\sqrt{-1}$$
$$\lambda_3 = -1.28 - .3\sqrt{-1}$$

and the general solution α of the differential equation (4.2) (1) is

$$\alpha = C_1 e^{-.127s} + C_2 e^{-1.28s} \cos(.3s + \varphi) \qquad (4.5)\ (6)$$

The first term represents the main contribution of gravity and corresponds to a non oscillating trajectory asymptotic to a vertical. The vertical trajectory is reached in a distance equal to about ten times the length of the projectile. Note that this conclusion depends on the magnitude of the velocity and holds only for the particular value of 25 ft/sec. assumed here. The second term represents a short wave oscillation about the trajectory and corresponds to the free motion as investigated in Chapter III. This free motion was found to be

$$z^* = C_1 e^{-1.56s} + C_2 e^{-1.12s} \qquad (4.5)\ (7)$$

Note that gravity influences this free motion to some degree by changing it to a damped oscillation. The oscillation is damped out within a travel equal to the bomb length.

Consider now a bomb with rather extreme characteristics

$\tau = 2$	$\varepsilon_p = .17$
$\varepsilon_t = .44$	$\varepsilon_2 = .44$
$\varepsilon_g = .05$	$\varepsilon_b = .17$
$i = .48$	$\mu = 5$
$\mu_g = 6.14$	

In this case the expression (4.5) (3) is positive and there is a critical speed. The value of this critical speed is

$$U_c = 5.2 \text{ ft/sec.}$$

and the period of oscillation at this speed is

$$T = 2.95 \text{ sec.}$$

These are the values in water. It is of interest of compare them with the critical speed and period of the same bomb in air. For the critical speed in air

$$U_c = 309 \text{ ft/sec.}$$

and the period

$$T = 2.8 \text{ sec.}$$

The period in air is remarkably close to that in water.

Conclusions

The general equations for the motion of the projectile and its trajectory contain the effect of gravity in the coefficients G_v and G_α of equations (4.1) (5). They are both functions of the Froude number $U/\sqrt{gl}$ and thereby contain implicitly the law of dynamics similarily for the motion under gravity. There are two stability criteria as expressed by (4.2) (5). The first one is independent of the Froude number and is usually satisfied for common type projectiles. Violation of this criterion leads to a divergence type of instability. The second depends on the Froude number and leads to two possibilities

(a) types of projectiles and fluid density where it is satisfied at all velocities

(b) types of projectiles and fluid density where there is a critical Froude number $U_c/\sqrt{gl}$ under which the motion is unstable.

The instability connected with this second criterion is of the nature of an oscillation of increasing amplitude about the vertical. The instability of a flat disc falling broadside-on belongs to this category.

For the aerial bomb case (b) is always realized, i.e. there is always a critical velocity. Instability in the numerical examples investigated is found to occur in a velocity range between zero to around 300 ft./sec. The period of the oscillation is of the order of two or three seconds and is independent of the density. A large value of the radius of gyration tends to increase the chance of instability. The condition that the terminal velocity be a critical velocity leads to a simple relation between the aerodynamic parameters of the bomb and is independent of the density. The drag coefficient necessary to satisfy this condition is quite large and can only be obtained with a parachute type tail of diameter two or three times that of the body of the bomb.

In water most usual designs fall in case (a), i.e. there is no critical velocity. In extreme cases the critical velocity exists but it will generally be very low. In the numerical example investigated this critical velocity is around 5 feet per second, while the period is about 3 seconds, i.e. of the order of that found in air.

In the example treated for a velocity around 20 feet per second in water the effect of a perturbation on the motion is found to be approximately the refraction of the free motion superposed on a gravity trajectory asymptotic to the vertical and determined by the initial refraction.

The type of instability analysed in this chapter plays an important role in all cases when bombs are released with small velocities relative to the air. This occurs for instance in retro-bombing and for bombs released from blimps or helicopters.

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5/26/43

List of Symbols

x, y, z	coordinates along body axes (moving with the body)
u, v, w	velocity components of the origin parallel to the instantaneous position of x, y, z.
p, q, r	components of angular velocity parallel to the instantaneous position of x, y, z.
T	kinetic energy of the fluid due to the motion of the body; also period of oscillation in Chapter IV.
A	longitudinal apparent mass of the fluid (x direction)
B	transversal apparent mass of the fluid (y direction)
R	rotational apparent mass of the fluid about OZ
A' P Q G'' etc.	coefficients in expression (1.1)(1).
A, B, C, D	in Chapter IV coefficients in the characteristic equation (4.2)(2)
X, Y, Z	components of the force exerted on the body by the fluid parallel to x, y, z.
$X_g, Y_g,$	components of the body weight along x, y.
L, M, N	moments of the forces exerted on the body by the fluid about x, y, z.
N_g	moment due to weight and buoyancy about the D. C.
$\dot{u}, \dot{v},$ etc.	derivatives of u, v, etc. with respect to time
U	large velocity component of the body in the x direction
U_c	critical velocity for stability
U_t	terminal velocity
α	angle of yaw (between x and x') In Chapter IV angle between the projectile axis and the vertical.
x' y'	axes with fixed directions.
X' Y'	force components of the fluid on the body parallel to the fixed directions x', y'.

u' v'	velocity components of the origin parallel to the fixed directions x', y'.
x_1	distance of virtual center of mass to the origin O.
v_1	velocity of the virtual center of mass parallel to y.
R_1	rotational apparent mass about the virtual center of mass.
k_1 k_2	inertia coefficient of the prolate ellipsoid for longitudinal and transversal apparent mass.
k'	inertia coefficient for the rotational apparent mass of the ellipsoid.
I_f	moment of inertia of virtual volume of fluid and moment of inertia of displaced fluid (in Chapter I)
V	volume of displaced fluid
ρ	mass of fluid per unit volume
ℓ	length of projectile
B_{vv} B_{uu} ... etc.	inertia coefficients (acceleration derivatives defined by (2.1)(1).
A_{vv} A_{uu} ... etc.	velocity derivatives defined by (2.1)(1)
D	drag
ω	circular frequency of oscillation
X'' Y'' N''	force components and moment of the fluid on the body when the origin is at O''
u'' v'' r''	velocity components when the origin is at O''
B''_{uu} B''_{vv} A''_{uu} A''_{vv}	stability derivatives when the origin is at O''
$\varepsilon\ell$	distance of O to O''
I	moment of inertia of the body about a transversal axis
k	spring constant

Y_t	transversal force due to tail action
$l\varepsilon_c$, $l\varepsilon_r$, $l\varepsilon_t$	distance of the origin to virtual center of mass (V.C.) rear point (R.P.) and tail center (T.C.) Figure (2,5)(3)
$\varepsilon_g l$, $\varepsilon_b l$	distance of C.G. and C.B. respectively from the D.C.
G_α , G_v	symbols defined by (4.1)(4) and representing the influence of the gravity in the basic equations.
v_c	transversal velocity at V.C.
v_r	transversal velocity at R.P.
τ	tail lift factor
C_L C_D C_m	lift, drag, and moment coefficients for airship models
	transversal (y) component of the velocity of the C. G.
I	moment of inertia of body about a transversal axis through the C. G.
a	distance of the C. G. of the body to the origin of the body axes.
m	mass of body
s =	$\frac{Ut}{l}$
t	time
$\frac{m}{\rho V} = \mu$	density of body relative to the fluid
$k = \sqrt{I_b/m}$	radius of gyration of the body about the dynamic center
$\mu_y = \mu + B_{vv}$	transversal density
$\mu_y i^2 = \mu \frac{k^2}{l^2} + B_{rr}$	
i	a radius of gyration expressed as a fraction of the length for the total transversal mass (body and fluid) about the dynamic center.

$C_{vv} = A_{vv}$

$C_{vr} = A_{vr} - \mu$

$C_{rv} = A_{rv}$

$C_{rr} = A_{rr} + \varepsilon_c B_{vv}$

$v^* = \frac{v}{U} = \text{yaw}$

$r^* = \frac{r\ell}{U}$

$E_1 = C_{vv} i^2 + C_{rr}$

$E_2 = C_{vv} C_{rr} - C_{rv} C_{vr}$

λ_1 , λ_2 characteristic exponents in the general solution of the equation of motion.

$\ell \varepsilon_p$ distance of the center of pressure to the dynamic center (positive when aft of the D.C.)

s_1 wave length of oscillation divided by

$F_v(s), F_r(s)$ arbitrary external forces acting on the body

$\eta = \frac{y'}{\ell}$ distance of the D.C. from the fixed axis x' measured as a multiple of the length

$\lambda = \frac{d}{ds}$ differential operator

$\Delta(\lambda)$ determinant (3,4)(7) $\Delta'(\lambda) = \frac{d}{d\lambda} \Delta(\lambda)$

$\Delta(0)$ value of $\Delta(\lambda)$ for $\lambda = 0$

$S(s)$ unit impulse function

$1(s)$ unit step function $\frac{d}{ds} 1(s) = S(s)$ $1(s) = 0$ for $s <$ [illegible] $1(s) = 1$ for $s >$ [illegible]

r_0 initial angular velocity about a transversal axis through the D.C. (for $s = 0$)

v_0 initial transversal velocity of the D.C. for $= 0$

$r_0^* = \frac{r_0 \ell}{U}$

$v_0^* = \frac{v_0}{U}$ initial yaw for $v_0 = 0$

α_0 initial angle between the body axis x and the fixed direction X'

$B_1\ B_2\ B_3\ B_4$	see (3.4)(15)
$b_1\ b_2\ b_3\ b_4$	coefficients in the expressions (3.4)(18) and (3.4)(19) for the asymptotic disturbed trajectories
η_∞	value of η for asymptotic trajectory

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